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RIDER

ANNIVERSARY

VOLUME

A series of papers presented on the occasion of his retirement by the friends of Paul Rider

RIDER ANNIVERSARY VOLUME

John V. Armitage

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AERONAUTICAL RESEARCH LABORATORIES - AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

OFFICE OF AEROSPACE RESEARCH

UNITED STATES AIR FORCE

February 21, 1963

Dedicated to Paul R. Rider

HARRY S. TRUMAN
INDEPENDENCE, MISSOURI

July 16, 1962

Dear Colonel:

I appreciate very much yours of the 6th, regarding my good friend and former neighbor, Dr. Paul R. Rider. His family and mine were next door neighbors in Independence for a long time.

Dr Rider has been one of the top notch citizens of Independence and Jackson County and, I can truthfully say, I wish we had more like him. He is a mathematical genius and has made a great contribution to the defense of the United States.

It gives me a great deal of pleasure to make this statement about my good friend and former neighbor, Dr. Paul R. Rider.

Sincerely yours,

Harry Truman

Colonel Robert E. Fontana, USAF
Commander
Aeronautical Research Laboratories
Office of Aerospace Research
United States Air Force
Wright-Patterson Air Force Base, Ohio

President Harry S. Truman provided this letter of dedication in response to a suggestion from Colonel Robert E. Fontana

RIDER

ANNIVERSARY

VOLUME

CONTENTS

THE DETERMINATION OF ROOTS IN A SYSTEM OF N POLYNOMIALS IN N VARIABLES

John V. Armitage

The method described in this paper reduces the problem of the determination of roots in a system of n polynomials in n variables to that of the determination of roots of several polynomials in one variable. Such systems occur in connection with problems involving chemical equilibria. As an alternate method of finding a root, one might consider an iterative scheme. Such schemes for the determination of roots by machine computation often cannot be used since suitable first approximations may not be available. The method developed in this paper depends upon an immediate extension of a classical theorem of the theory of equations to the effect that a necessary and sufficient condition that two polynomials in one unknown have a root in common is the vanishing of Sylvester's eliminant of their coefficients. This theorem has been extended in the literature to n equations in n-1 unknowns (Reference [1]).

Let us first consider the case of two polynomials in two variables. Given the linearly independent polynomials

$$P_1(x,y) = U_n(x)y^n + U_{n-1}(x)y^{n-1} + \cdots + U_o(x) \tag{1}$$

$$P_2(x,y) = V_m(x)y^m + V_{m-1}(x)y^{m-1} + \cdots + V_o(x) \tag{2}$$

where the U's and V's are polynomials in x.

Define the following polynomial in x which corresponds to Sylvester's eliminant:

$$\Delta(x) = \begin{vmatrix} U_n(x) & U_{n-1}(x) & \cdots & U_o(x) & 0 & 0 & \cdots & 0 \\ 0 & U_n(x) & \cdots & U_1(x) & U_o(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & U_n(x) & \cdots & U_o(x) \\ V_m(x) & V_{m-1}(x) & \cdots & V_o(x) & 0 & 0 & \cdots & 0 \\ 0 & V_m(x) & \cdots & V_1(x) & V_o(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & V_m(x) & \cdots & V_o(x) \end{vmatrix} \tag{3}$$

The following theorem will now be proved: The necessary condition for the existence of roots x_1, y_1 which simultaneously satisfy $P_1(x,y)$ and $P_2(x,y)$ is that x_1 be a root of $\Delta(x)$.

<u>Proof</u>: Regard the $U_i(x)$ and $V_i(x)$ for the moment merely as coefficients of polynomials in the variable y. Any value of y which is simultaneously a root of $P_1(x,y) = 0$ and $P_2(x,y) = 0$ must also be a root of

$$\begin{aligned}
yP_1(x,y) &= 0 \\
y^2P_1(x,y) &= 0 \\
&\vdots \\
y^{m-1}P_1(x,y) &= 0 \\
yP_2(x,y) &= 0 \\
&\vdots \\
y^{n-1}P_2(x,y) &= 0
\end{aligned} \tag{4}$$

If we consider $y^1, y^2, \cdots, y^{n+m-1}$ as independent unknowns we have arrived at a linear system of n+m equations. In order that such a system be consistent it is necessary that the determinant of its coefficients vanish; i.e., that $\Delta(x) = 0$.

The condition $\Delta(x) = 0$ is also a sufficient condition for the existence of roots x_i, y_i provided that a root x_1 of $\Delta(x) = 0$ is not simultaneously a root of $U_n(x)$ and $V_m(x)$.

<u>Proof</u>: Suppose x_1 is a root of $\Delta(x)$ but not a root of $U_n(x)$. $\Delta(x_1) = 0$ implies that there is a linear dependence in system (4). That is, there exists an identity in y of the form

$$P_1(x_1, y)\ [a_{m-1}y^{m-1} + \cdots + a_o] + P_2(x_1, y)\ [b_{n-1}y^{n-1} + \cdots + b_o] \equiv 0 \tag{5}$$

where the a_i and b_i are constants not all equal to zero.

Case 1. At least one of the a_i not equal to zero: This implies that at least one of the b_i is not zero. If, on the contrary, $b_i = 0$ for all i, then in consideration of (5), since the polynomial $a_{m-1}y^{m-1} + \cdots a_o \not\equiv 0$, it follows that $P_1(x_1, y) \equiv 0$. This contradicts the assumption that $U_n(x_1) \not\equiv 0$. Now we will show that at least one of the roots of $P_1(x_1, y)$ is also a root of $P_2(x_1, y)$.

First, $P_2(x_1, y)$ is a polynomial of degree r in y where $1 \leqq r \leqq m$. The fact that $r > 0$ is shown as follows: The assumption $U_n(x_1) \not\equiv 0$ implies that $P_1(x_1, y)$ is a polynomial in y of degree n. It follows from identity (5) that

$$P_2(x_1, y)\ [b_{n-1}y^{n-1} + \cdots + b_o] \tag{6}$$

is a polynomial of degree greater than or equal to n. This is impossible unless $r > 0$.

Next let us assume that there are no roots of $P_1(x_1, y)$ which are also roots of $P_2(x_1, y)$. Consider polynomial (6). This has degree less than or equal to $n + r - 1$.

Let $y_1, \ldots, y_n$ be the roots of $P_1(x_1, y)$. By identity (5) $y_1, \ldots, y_n$ are also roots of (6). But the r roots of $P_2(x_1, y)$ are also roots of (6). Hence (6) is a polynomial with at least $n + r$ roots, a contradiction since the degree of (6) is not greater than $n + r - 1$. It follows that at least one root of $P_1(x_1, y)$ is also a root of $P_2(x_1, y)$.

Case 2. At least one of the b_i is non-zero: In this case either

(a) At least one of the a_i of equation (5) is non-zero. In this case the discussion in Case 1 applies.

(b) $a_i = 0$ for all i. Then, by equation (5)

$$P_2(x_1, y)\ [b_{n-1}y^{n-1} + \cdots + b_o] \equiv 0$$

But since at least one of the b_i is non-zero, this implies that $P_2(x_1, y) \equiv 0$. Hence, any root y_i of $P_1(x_1, y)$ will be suitable.

We now proceed to establish a necessary condition for the existence of roots in a system of n polynomials in n unknowns. The proof will be by induction.

Suppose that the following proposition is true:

Given:

$$\left.\begin{aligned} P_1(x_1, x_2, \ldots, x_{n-1}) &= 0 \\ P_2(x_1, x_2, \ldots, x_{n-1}) &= 0 \\ &\vdots \\ P_{n-1}(x_1, x_2, \ldots, x_{n-1}) &= 0 \end{aligned}\right\} \tag{7}$$

Define

$$\left.\begin{aligned} &\Delta_{1,1}(x_1, x_2, \ldots, x_{n-2})^* \\ &\Delta_{1,2}(x_1, x_2, \ldots, x_{n-2}) \\ &\qquad\vdots \\ &\Delta_{1,n-1}(x_1, x_2, \ldots, x_{n-2}) \end{aligned}\right\} \tag{8}$$

where $\Delta_{1,k}(x_1, x_2, \ldots, x_{n-2})$ is the eliminant of x_{n-1} from P_1 and P_{k+1}. Further, similarly define

$$\left.\begin{aligned} &\Delta_{2,1}(x_1, x_2, \ldots, x_{n-3}) \\ &\qquad\vdots \\ &\Delta_{2,n-2}(x_1, x_2, \ldots, x_{n-3}) \end{aligned}\right\} \tag{9}$$

* One might use $\Delta_{1,1}^{(n-1)}(x_1, x_2, \ldots, x_{n-2})$ as an alternate notation. Here the superscript represents the order of the system. However, if one notes that the order of the original system equals the sum of the first subscript of Δ plus the subscript of the last listed variable then the superscript is not necessary.

to be eliminants of x_{n-2} from system (8). Continue this process until we obtain finally

$$\Delta_{n-2,1}(x_1) \tag{10}$$

where, at each step in the process, we have obtained a system of one fewer equations with one fewer unknowns. Then the necessary condition that there exist roots which simultaneously satisfy system (7) is that $\Delta_{n-2,1}(x_1) = 0$.

<u>Proof</u>: Suppose that we are given the system

$$\left.\begin{aligned} P_1(x_1,\ x_2,\ \ldots,\ x_n) &= 0 \\ P_2(x_1,\ x_2,\ \ldots,\ x_n) &= 0 \\ &\vdots \\ P_n(x_1,\ x_2,\ \ldots,\ x_n) &= 0 \end{aligned}\right\} \tag{11}$$

where the P_i are expressed in powers of x_n. If we regard the coefficients of x_n as undetermined quantities, in order that there exist an x_n consistent between P_1 and P_2 it is necessary that

$$\Delta_{1,1}(x_1,\ x_2,\ \ldots,\ x_{n-1}) = 0$$

Similarly one is led to

$$\Delta_{1,2}(x_1,\ x_2,\ \ldots,\ x_{n-1}) = 0,\ \ldots,\ \Delta_{1,n-1}(x_1,\ x_2,\ \ldots,\ x_{n-1}) = 0$$

But this represents a system of $n-1$ equations in $n-1$ unknowns. Hence, by the inductive assumption it is necessary that $\Delta_{n-1,1}(x_1) = 0$. We have shown that this procedure is true for $n = 2$. Hence, by induction it is true for all n.

The procedure for the determination of roots in a system of n polynomials is now clear. For simplicity it will be demonstrated on a system of three polynomials in three unknowns.

Given the system

$$P_1(x,y,z)$$
$$P_2(x,y,z) \qquad (12)$$
$$P_3(x,y,z)$$

form eliminants $\Delta_{1,1}(x,y)$ and $\Delta_{1,2}(x,y)$. From this system form the eliminant $\Delta_{2,1}(x)$. Find the roots $x_1, x_2, \ldots, x_p$ of $\Delta_{2,1}(x)$. Substitute a root, say x_1 into $\Delta_{1,1}(x,y)$ and find the roots of $\Delta_{1,1}(x_1, y)$. These roots may then be substituted into $\Delta_{1,2}(x,y)$. A pair of values, say x_1, y_1 which satisfies $\Delta_{1,2}(x,y)$ is then substituted into $P_1(x,y,z)$ and roots z_1 of $P_1(x_1, y_1, z)$ are found. These roots can be checked to determine whether they satisfy $P_2(x_1, y_1, z)$ and $P_3(x_1, y_1, z)$. By using this procedure one is assured that all possible roots of the system will be found.

This method can be applied to the problem of the determination of the complex roots of a polynomial. For example, given the cubic polynomial

$$x^3 + 3a_2x^2 + 3a_1x + a_0 \qquad (13)$$

If we set $x = y + iz$ we obtain:

$$y^3 - 3yz^2 + 3a_2y^2 - 3a_2z^2 + 3a_1y + a_0 + i(3y^2z - z^3 + 6a_2yz + 3a_1z) \qquad (14)$$

A necessary and sufficient condition that $y_1 + iz_1$ be a root of (14) is that y_1 and z_1 simultaneously satisfy the equations

$$3(y + a_2)z^2 - y^3 - 3a_2y^2 - 3a_1y - a_0 = 0 \qquad (15a)$$

$$z[z^2 - 3(y^2 + 2a_2y + a_1)] = 0 \qquad (15b)$$

The factor z of equation (15b) can be neglected since $z = 0$ is the condition

that equation (13) have a real root. We form Sylvester's eliminant of system (15) and obtain the polynomial

$$[8y^3 + 24a_2y^2 + 6(a_1 + 3a_2^2)y + 9a_1a_2 - a_0]^2 = 0 \tag{16}$$

Hence the real roots of

$$8y^3 + 24a_2y^2 + 6(a_1 + 3a_2^2)y + 9a_1a_2 - a_0 = 0$$

will give the real parts of the roots of equation (13): the complex parts may then be determined in an obvious way from system (15).

Reference

[1] Turnbull, H. W., Theory of Equations, Interscience Publishers, Inc. (1957).

STATISTICAL ESTIMATES FOR THE RANDOM ERRORS IN TABULAR DIFFERENCES

Gertrude Blanch

and

Donald S. Clemm

1. Background

One of the most powerful tools for testing the accuracy of entries, in a systematic tabulation of a well-behaved function, is to examine successive differences of the function. A well-planned tabulation is *interpolable* by an interpolation formula of fairly low degree. Thus five-place trigonometric tables of $\sin x$ and $\cos x$ are linearly interpolable. Many tables of the higher mathematical functions are interpolable by formulas of degree 2 or 4, and in practically all cases of interest, of degree no higher than 8.

Interpolability implies that the interval, h, between successive arguments is constant over a considerable range of the table. Moreover, h must be sufficiently small to insure that the successive differences of the function tabulated fall off numerically, and that differences of a sufficiently high order, say n, when carried to a fixed number of decimals, approach zero.

The n^{th} advancing difference of $f(x)$, at $x = x_0$, can be defined as follows:

$$\Delta^n f(x_0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x_{n-k}) \quad ; \quad \binom{n}{k} = \frac{n!}{k!\ (n-k)!} \tag{1.00}$$

In the above, it is assumed that $x_j = x_0 + jh$, where h is the interval of tabulation.

If, now, the entries $f(x)$ are rounded numbers, let $f^*(x)$ represent the true value, so that

$$f^*(x_j) = f(x_j) + \delta_j \tag{1.01}$$

The *rounding errors*, δ_j, can often be considered as mutually independent, rectangularly distributed variates, ranging in magnitude between bounded limits, say $-\epsilon$ and $+\epsilon$. Suppose that the n^{th} difference of $f^*(x)$, for some n, is known to be zero, to the number of decimals carried. (What follows has meaning only if we assume

that all entries under consideration are carried to the same number of decimal places, or other radix digits.) In that case, $\Delta^n f(x_j)$ will consist, essentially, of the δ_j, namely

$$\Delta^n f(x_j) = \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \delta_{n-k} = u_n(\delta) \tag{1.02}$$

say, where δ stands for the various values, considered random, that the δ_{n-k} take on.

The study of the values that $u_n(\delta)$ assumes as x takes on a sequence of values x_0, x_0+h, ..., x_0+sh, is under consideration. The *range* of $u_n(\delta)$ can be simply stated. Thus, if the δ_j are rectangularly distributed between $\pm\epsilon$, then

$$u_1(\delta) = \delta_j - \delta_{j-1}, \qquad j = 1, 2, \ldots$$

will range between $\pm 2\epsilon$. By induction, it can be verified that the range of $u_n(\delta)$ is given by $-2^n\epsilon \leq u_n(\delta) \leq 2^n\epsilon$. We now pass to the more difficult problem of obtaining the distribution of $u_n(\delta)$ over its range.

2. Properties of Sums of Variates

Let $f_{0,1}(y)$ denote the probability that δ lies between y and $y + \Delta y$. Since δ is rectangularly distributed,

$$f_{0,1}(y) = \begin{cases} 1/2\epsilon, & \text{if } -\epsilon \leq y \leq \epsilon \\ 0 \quad, & \text{otherwise} \end{cases} \tag{2.01}$$

Let $f_{k,n}(y)$ denote the probability that δ lies between $\binom{n}{k}$ y and $\binom{n}{k}$ $(y+\Delta y)$. Moreover, since $\binom{n}{k}$ is a constant, it follows that the distribution of $\binom{n}{k}\delta$ is given by

$$f_{k,n}(y) = \begin{cases} 1/\binom{n}{k} 2\epsilon, & \text{if } -\binom{n}{k}\epsilon \leq y \leq \binom{n}{k}\epsilon \\ 0 \quad, & \text{otherwise} \end{cases} \tag{2.02}$$

The δ_{n-k} in (1.02) are symmetrically distributed between $-\epsilon$ and $+\epsilon$; therefore the factors $(-1)^k$ in (1.02) play no role; thus $u_n(\delta)$ has the same distribution as

$$v_n(\delta) = \sum_{k=0}^{k=n} \binom{n}{k} \delta_{n-k}$$

It is known that the frequency density function of the sum of the $n+1$ independent variates, $\delta_0, n\delta_1, \ldots, \binom{n}{k} \delta_k, \ldots, \delta_n$ is the product of their density functions. Hence, the product could have been written down immediately if the expression for $f_{k,n}(y)$ in (2.02) were the same over the entire range of y. However, three distinct segments are involved, and so (2.02) does not lend itself to mathematical manipulations, for large values of n.

Fortunately it is possible to obtain an alternative definition for (2.02) by the use of the notion of *characteristic functions*. This notion was used by Lowan and Laderman (Reference [2]), who gave the distribution of $u_n(\delta)$ for $n = 1, 2, 3$. These authors derived the exact expressions, as polynomial segments, that the functions consist of. An examination of some of the formulas involved, given in Section 3, shows that the amount of labor in generating the functions becomes prohibitive for n greater than 3, if one is limited to hand computations. This explains why no further tabulation has been supplied since Reference [2]. For high-speed computers, on the other hand, the work is still tractable, and we here supply the frequency curves and cumulative areas for $n \leq 8$. Results are given in tabular form, rather than as polynomial expressions.

3. The Characteristic Function

Here we follow, in essentials, the procedure used in Reference [2]. If $f(x)$ is a frequency distribution function, its *characteristic function*, $g(t)$, is defined by

$$g(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \tag{3.01}$$

(See Reference [1] for a derivation.) In turn, if $g(t)$ is known, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} g(t)\, dt \tag{3.02}$$

If $f(x)$ is the distribution function of (2.02), then one may substitute the right-hand side of (2.02) for $f(x)$ in (3.01) and then integrate between $-\binom{n}{k}\epsilon$ and $\binom{n}{k}\epsilon$. The result is

$$g_{k,n}(t) = \frac{\sin\left[\epsilon \binom{n}{k} t\right]}{\epsilon \binom{n}{k} t} \tag{3.03}$$

Thus, from (3.02) and (3.03), an alternative expression for $f_{k,n}(y)$ is

$$f_{k,n}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ity} \sin\left[\epsilon \binom{n}{k} t\right]}{\epsilon \binom{n}{k} t}\, dt \tag{3.04}$$

It is a noteworthy fact that (3.04) is a precise expression for $f_{k,n}(y)$ at any point y. In contrast, equation (2.02) requires three disjoint expressions, depending on whether y is positive or negative, and whether y is numerically less than or greater than $\binom{n}{k}\epsilon$. It is this fact that makes (3.04) useful in deriving the general expression for $u_n(\delta)$, when the latter is defined by (1.02). It will be

instructive to verify, at this stage, that (3.04) in fact coincides with (2.02). This study will point to some basic properties of the integrals of type (3.04), which will be useful for more complicated expressions.

If α is finite and real, the value of $\sin \alpha t / \alpha t$ is bounded for all real t, and is unaltered if t is replaced by $-t$. Again, $\sin ty$ changes sign with t; hence the imaginary component of (3.04) vanishes, as it should, and (3.04) can be replaced by the real integral

$$f_{k,n}(y) = \frac{1}{\pi} \int_0^\infty \frac{\cos ty \sin \left[\epsilon \binom{n}{k} t \right]}{\epsilon \binom{n}{k} t} \, dt$$

Let $\epsilon t = w$. The above becomes

$$f_{k,n}(y) = \frac{1}{\pi\epsilon} \int_0^\infty \frac{\cos \left(w \frac{y}{\epsilon} \right) \sin \left[\binom{n}{k} w \right]}{\binom{n}{k} w} \, dw \tag{3.05}$$

The following four well-known relations will be useful:

$$\int_0^\infty \frac{\sin aw}{w} \, dw = \begin{cases} \frac{1}{2}\pi, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \\ -\frac{1}{2}\pi, & \text{if } a < 0 \end{cases} \tag{3.06}$$

$$\sin a \cos b = \frac{1}{2} [\sin (a-b) + \sin (a+b)] \tag{3.07}$$

$$\sin^2 a = \frac{1}{2} (1 - \cos 2a) \tag{3.08}$$

$$\sin a \sin b = \frac{1}{2} [\cos (a-b) - \cos (a+b)] \tag{3.09}$$

Using (3.06) and (3.07) in (3.05), one obtains, for $y \geq 0$,

$$f_{k,n}(y) = \begin{cases} 1/2\epsilon \binom{n}{k} & , \text{ if } 0 \leq y < \epsilon \binom{n}{k} \\ 1/4\epsilon \binom{n}{k} & , \text{ if } y = \epsilon \binom{n}{k} \\ 0 & , \text{ if } y > \epsilon \binom{n}{k} \end{cases} \tag{3.10}$$

It is seen that (3.10) agrees with (2.02), except for the point $y = \epsilon \binom{n}{k}$; this is a point of discontinuity of (2.02) as well. Agreement of (2.02) with (3.10) is therefore in the 'almost everywhere' sense. Note that (3.05) also reflects the fact that $f_{k,n}(y)$ is an even function of y, and hence y may be replaced by $|y|$ on the right-hand side of (3.10).

The characteristic function of a sum of mutually independent variates is the product of their respective characteristic functions. Let $G_n(t)$ be the characteristic function associated with $u_n(\delta)$ in (1.02). From (3.04),

$$G_n(t) = \prod_{k=0}^{n} \frac{\sin \epsilon \binom{n}{k} t}{\epsilon \binom{n}{k} t} \tag{3.11}$$

4. Derivation of Expressions for the Distribution Functions

Let $\phi_n(y)$ denote the probability that $u_n(\delta)$ in (1.02) lies between y and $y + \Delta y$. Using the principle that led to (3.05), and considering (3.11), one obtains

$$\phi_n(y) = \frac{1}{\pi\epsilon \prod_{k=0}^{n} \binom{n}{k}} \int_0^\infty \frac{\cos\left(w \frac{y}{\epsilon}\right) \prod_{k=0}^{n} \sin\left[\binom{n}{k} w\right] dw}{w^{n+1}} \tag{4.01}$$

Let

$$F_n(w) = \cos\left(w\frac{y}{\epsilon}\right) \prod_{k=0}^{n} \sin\left[\binom{n}{k} w\right] \tag{4.02}$$

At $w = 0$, $F_n(w)/w^{n+1}$ is finite, and moreover

$$\lim_{w \to 0} [F_n^{(k)}(w)/w^{n-k}] = 0 \ , \quad k \leq n$$

Similarly

$$\lim_{w \to \infty} [F_n^{(k)}(w)/w^{n-k}] = 0 \ , \quad k \leq n - 1$$

Hence, integrating by parts,

$$\int_0^\infty \frac{F_n(w)\ dw}{w^{n+1}} = \int_0^\infty \frac{F_n'(w)\ dw}{nw^n}$$

Repeating the operation k times, one obtains

$$\int_0^\infty \frac{F_n(w)\ dw}{w^{n+1}} = \int_0^\infty \frac{F_n^{(k)}(w)\ dw}{n(n-1)\ \cdots\ (n-k+1)\, w^{n+1-k}}$$

Setting $k = n$ in the above,

$$\int_0^\infty \frac{F_n(w)\ dw}{w^{n+1}} = \int_0^\infty \frac{F_n^{(n)}(w)\ dw}{n!\ w} \tag{4.03}$$

With the aid of the trigonometric identities (3.07) through (3.09), it is easy to

show that $F_n^{(n)}(w)$ can be decomposed into a sum of sine terms, both when n is even and when n is odd, of the form

$$F^{(n)}(w) = \sum_{k=0}^{M} d_k \sin(a_k w) = \sum_{k=0}^{M} d_k^* \sin(\alpha_k w) , \qquad \alpha_k \geq 0 \tag{4.04}$$

In the preceding, M and the a_k can be determined, and we set $d_k^* = -d_k$ if a_k is negative; $d_k^* = d_k$ if a_k is positive or zero. The α_k will, of course, involve y. Since $\phi_n(y)$ is an even function of y, it will be sufficient to consider values of $y \geq 0$. If the a_k, d_k, and M are known, then substituting (4.04) into (4.01) and using (3.06), one obtains

$$\phi_n(y) = \frac{1}{2\pi K_n} \frac{1}{n!} \sum_{k=0}^{M} d_k^* \; ; \qquad K_n = \prod_{k=0}^{n} \binom{n}{k} \tag{4.05}$$

In application, one is also interested in the following question: What is the probability, $P_n(y)$, that $u_n(\delta)$ will lie between $-z$ and $+z$? Clearly

$$P_n(z) = \int_{-z}^{z} \phi_n(y)\, dy = 2 \int_{0}^{z} \phi_n(y)\, dy$$

From (4.01),

$$P_n(z) = \frac{2}{\pi\epsilon} \frac{1}{K_n} \int_0^z dy \int_0^\infty \frac{\cos\left(w \frac{y}{\epsilon}\right) \prod_{k=0}^{n} \sin\left[\binom{n}{k} w\right] dw}{w^{n+1}} \tag{4.06}$$

Let $z = x\epsilon$. The above becomes, after a simple transformation on y,

$$P_n(x\epsilon) = \frac{2}{\pi K_n} \int_0^x dt \int_0^\infty \frac{\cos (wt) \prod_{k=0}^{n} \sin \left[\binom{n}{k} w\right]}{w^{n+1}} dw \qquad (4.07)$$

It is permissible to interchange limits of integration. Doing so, and integrating with respect to t, one obtains, finally

$$P_n(x\epsilon) = \frac{2}{\pi K_n} \int_0^\infty \frac{\sin (xw) \prod_{k=0}^{n} \sin \left[\binom{n}{k} w\right]}{w^{n+2}} dw, \quad x \geq 0 \qquad (4.08)$$

Using the method applied to the determination of $\phi_n(y)$, it is possible to write

$$P_n(x\epsilon) = \frac{2}{\pi K_n} \int_0^\infty \frac{Q^{(n+1)}(w)\ dw}{w(n+1)!} \qquad (4.09)$$

where

$$Q_n(w) = \sin xw \prod_{k=0}^{n} \sin \left[\binom{n}{k} w\right] \qquad (4.10)$$

At this point we shall go into somewhat more detail in examining the terms of $Q_n(w)$, when the latter is decomposed into a sum of sines or cosines. The factors entering into the product that multiplies $\sin xw$ in (4.10) will consist of one of the following two forms:

$$\prod_{k=0}^{n} \left[\sin \binom{n}{k} w\right] = \Sigma d_j \sin c_k w \ ; \quad \text{or} \quad \Sigma d_j \cos c_k w \qquad (4.11)$$

where the c_j are even integers. These c_j are, in fact, the sums of all possible combinations of $\binom{n}{k}$, for all k, taken one, two, ..., n at a time. Such sums are necessarily even integers or zero, whether n is even or odd. If n is even, then $\binom{n}{k}$ is even. If n is odd, then every $\binom{n}{k}$ is paired with $\binom{n}{n-k}$ When the terms of (4.11) are multiplied by $\sin xw$, there results an expression for $Q_n(w)$ of the form

$$Q_n(w) = \begin{cases} \Sigma b_j \ [\sin (x+c_k)w], & \text{if } n \text{ is odd} \\ \\ \Sigma b_j \ [\cos (x+c_k)w], & \text{if } n \text{ is even} \end{cases} \tag{4.12}$$

The differentiation of each term of (4.12) $n+1$ times yields an expression of the form

$$Q_n^{(n+1)}(w) = \Sigma b_j^* \ (x+c_k)^{n+1} \sin (x+c_k)w \tag{4.13}$$

In the above, $b_j^* = \pm b_j$. It is to be noted that the decomposition is always into a sum of sine-terms, whether n is even or odd. A similar remark applies to the decomposition of the terms involved in the expressions for $\phi_n(y)$. When the terms of (4.13) are substituted into the integral of (4.09), there result polynomials in x, of degree no higher than $n+1$, in the expression for $P_n(x)$.

5. The Vertices of the Distribution

The vertices of the distribution can be determined from the expression for either $\phi_n(y)$ or $P_n(x)$. Since more detail was given for the latter in the foregoing discussion, let us consider the expression for $P_n(x)$. When (4.13) is substituted into (4.09), it is clear that a term arising out of (4.13) will contribute

the following:

$$b_j^* \, (x+c_k)^{n+1} \int_0^{\infty} \frac{\sin\,[(x+c_k)w]}{w}\, dw = \begin{cases} \frac{\pi}{2}\, b_j^* \, (x+c_k)^{n+1}, & \text{if } (x+c_k) > 0 \\ 0\ , & \text{if } (x+c_k) = 0 \\ -\frac{\pi}{2}\, b_j^* \, (x+c_k)^{n+1}, & \text{if } (x+c_k) < 0 \end{cases} \tag{5.01}$$

Each such term therefore contributes a polynomial in x, of degree no higher than $n+1$. Thus as $x+c_k$ passes through zero, this set of terms in the polynomial in x will change in sign; in other words, $x_0 = -c_k$ will be a *vertex*, since the polynomial expression will be different in sign depending on whether x is greater than or less than x_0. From symmetry, c_k will also be a vertex. For $n = 3$, all the even integers $\leq 2^3$ are vertices, including 0. For $n = 4$, all the even integers $\leq 2^4$, excluding 0 and 10, are vertices. The pattern becomes more complicated as n increases. Both $\phi_n(x\epsilon)$ and $P_n(x\epsilon)$ are continuous. At a vertex, some derivative of $\phi_n(x\epsilon)$ is not uniquely defined. However, at the tabular interval given here, the functions (to 5 decimal places) are so smooth that the vertices cannot be discerned from the differences of the entries. An examination of some of the entries at a coarser interval, and to more decimal places, does indeed reveal the vertices.

6. Comparison with the Normal Frequency Distribution

The central limit theorem gives assurance that

$$\lim_{n\to\infty} \phi_n(x\epsilon) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}(x/\sigma)^2} \tag{6.01}$$

$$\lim_{n\to\infty} P_n(x\epsilon) = 2 \int_0^{x\epsilon} \phi_n(t\epsilon)\, d(t\epsilon) \tag{6.02}$$

where

$$\sigma = \sigma(\epsilon)\sqrt{\sum_{k=0}^{n}\binom{n}{k}^2} = \epsilon\sqrt{1/3}\sqrt{\sum_{k=0}^{n}\binom{n}{k}^2} \tag{6.03}$$

In particular, if $x\epsilon = \lambda\sigma$,

$$\lim_{n\to\infty} \phi_n(\lambda\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\lambda^2} \tag{6.04}$$

$$\lim_{n\to\infty} P_n(\lambda\sigma) = \frac{2}{\sqrt{2\pi}}\int_0^{\lambda} e^{-\frac{1}{2}t^2}dt \tag{6.05}$$

It is to be noted that at $\lambda = 0$, $\lim\limits_{n\to\infty} \phi_n(0) = 1/\sigma\sqrt{2\pi}$.

The question arises: How closely do the tabular values approach the normal distribution when $n = 8$? Schedule A lists representative values of the distribution, along with the corresponding ones from the normal curve and its area function. The agreement is to within about 5 percent error and comparison of results for successively smaller values of n shows that the distributions do not approach the normal one very rapidly. It is therefore hoped that the present tabulation will serve a useful purpose in providing a knowledge of the exact pattern of random errors in tabular differences through the eighth.

The graphs of the distribution show the shapes of the various curves.

7. Method of Computation

An IBM 7090 Fortran program, using double-precision arithmetic, was used to decompose $F_n^{(n)}(w)$ of (4.03) and $Q_n^{(n+1)}(w)$ in (4.09) into a sum of sines, for

every assigned value of x, and to generate the resulting integrals. The results for $n \leq 3$ were compared with the theoretical results available in Reference [2]. The 5-place rounded entries were also differenced.

8. Acknowledgments

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Schedule A

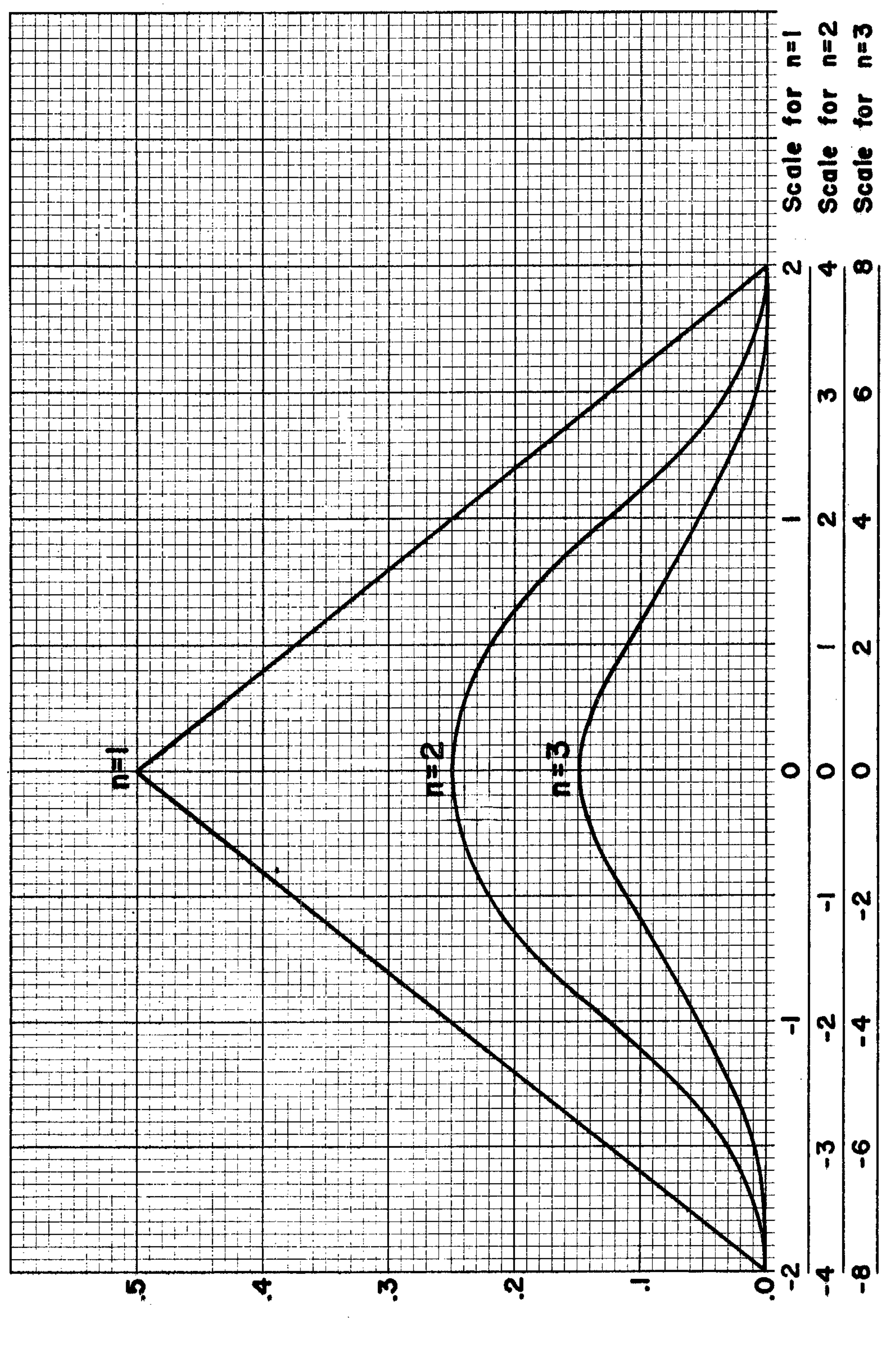
Comparisons for n = 8

x	$\frac{1}{\sigma} x$	Table Ordinate	Table Integral	Normal Distribution $\frac{1}{\sigma} z(x/\sigma)$	Normal Distribution $\alpha(x/\sigma)$
0	.0	.00582	.0	.00609	.0
1	.01527	.00582	.01163	.00609	.01218
10	.15268	.00576	.11599	.00602	.12135
50	.76338	.00456	.53825	.00455	.55476
75	1.14507	.00330	.73547	.00316	.74782
100	1.52676	.00207	.86904	.00190	.87318
125	1.90845	.00109	.94648	.00099	.94367
150	2.29014	.00045	.98342	.00044	.97799
175	2.67183	.00013	.99667	.00017	.99246
200	3.05352	.00002	.99969	.00006	.99774
225	3.43521	.00000	1.00000	.00002	.99941

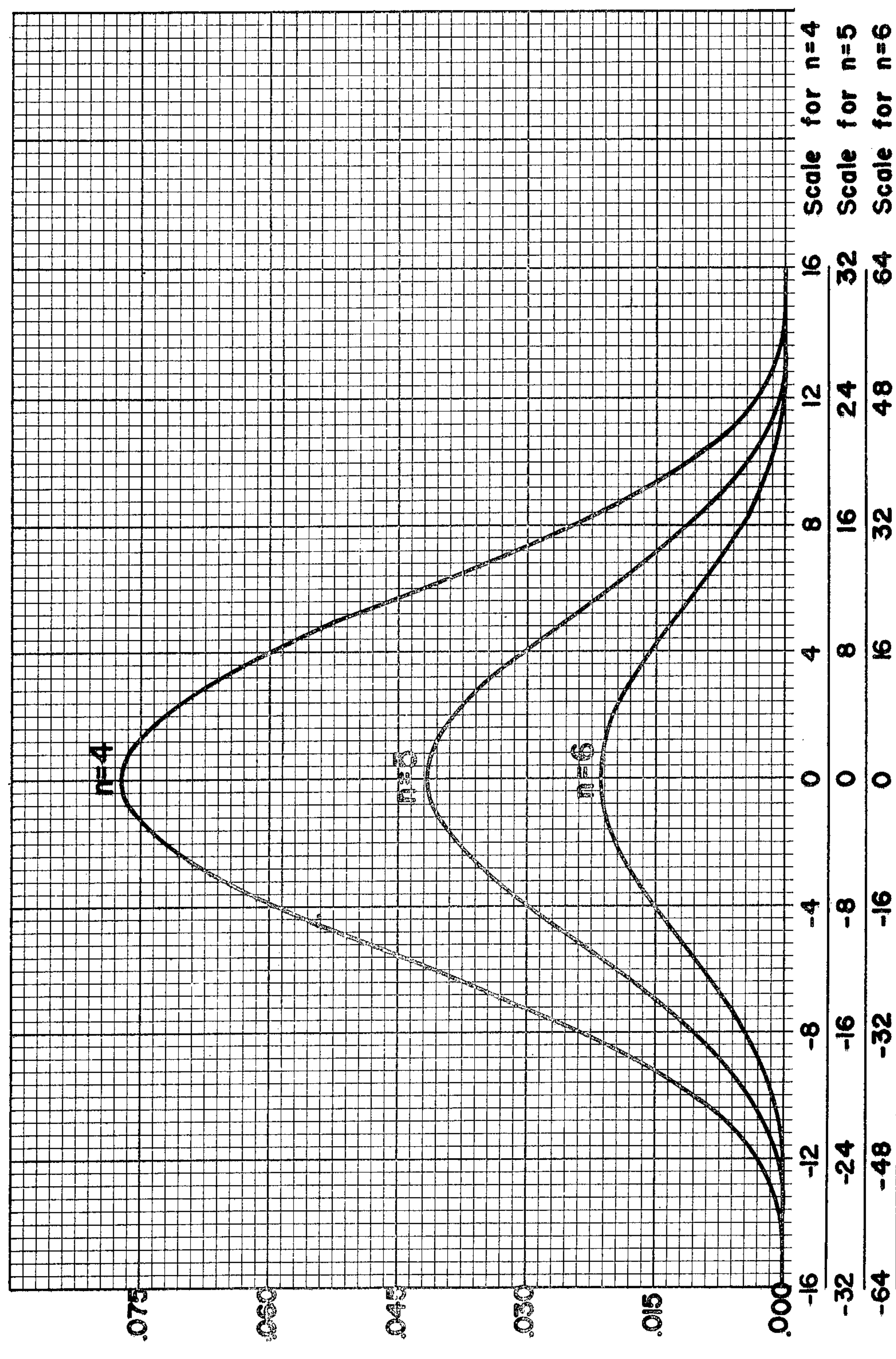
$$\sigma = \sqrt{\frac{1}{3} \sum_{k=0}^{8} \binom{8}{k}^2} = 65.4981$$

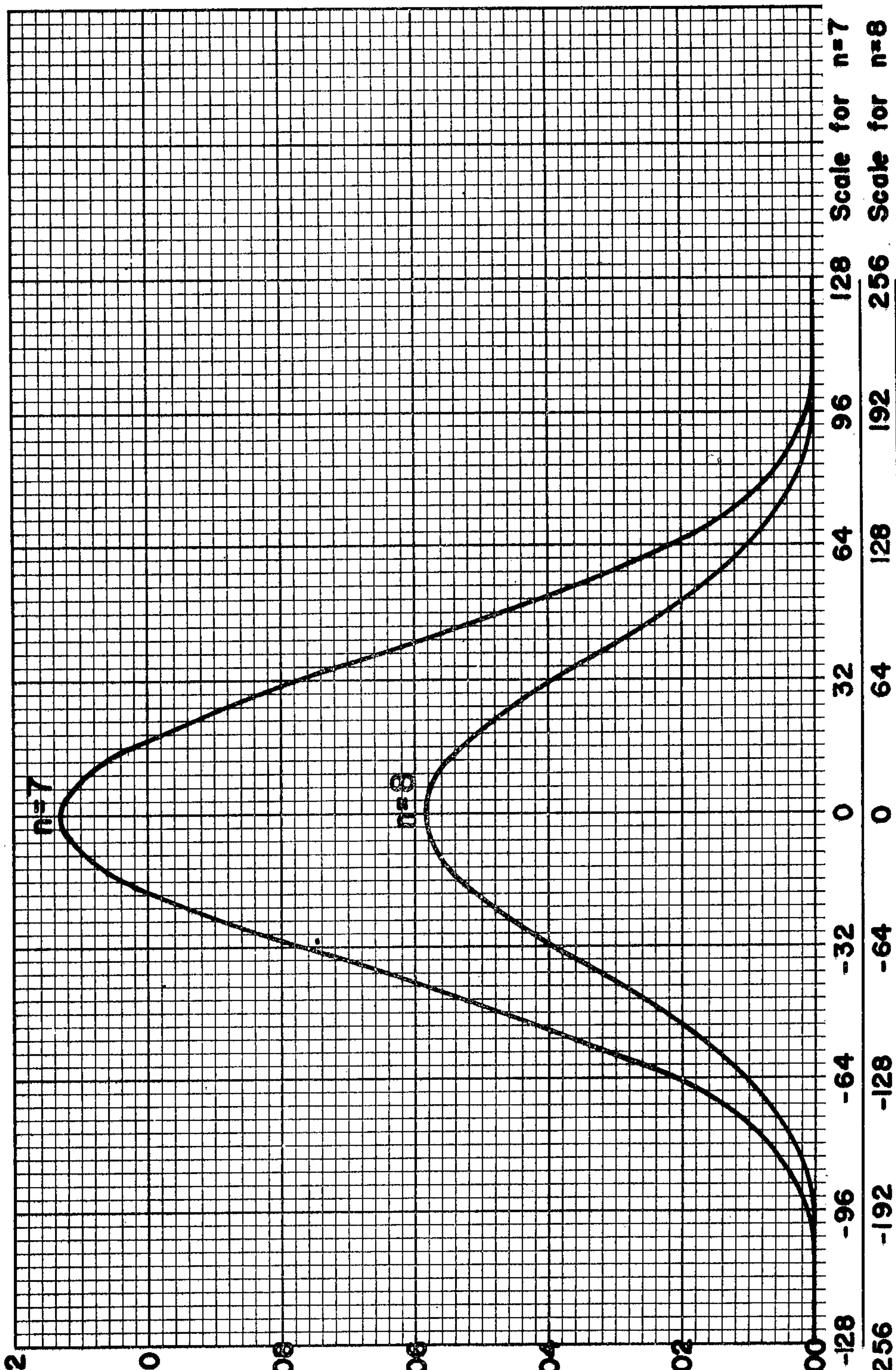
$$z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}$$

$$\alpha(x) = \int_{-x}^{x} z(t)\, dt$$



n=1
n=2
n=3
.0
.1
.2
.3
.4
.5
-2 -1 0 1 2
-4 -3 -2 -1 0 1 2 3 4
-8 -6 -4 -2 0 2 4 6 8
Scale for n=1
Scale for n=2
Scale for n=3


n=4
n=5
n=6
.075
.060
.045
.030
.015
.000
-16 -12 -8 -4 0 4 8 12 16 Scale for n=4
-32 -24 -16 -8 0 8 16 24 32 Scale for n=5
-64 -48 -32 -16 0 16 32 48 64 Scale for n=6



n=7
n=8
.012
.010
.008
.006
.004
.002
.000
-128 -96 -64 -32 0 32 64 96 128 Scale for n=7
-256 -192 -128 -64 0 64 128 192 256 Scale for n=8

Table of $\epsilon\phi_n(x\epsilon)$ and $P_n(x\epsilon)$

Definitions

Let

$$u_n = \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \delta_k$$

where the δ_k are rectangularly distributed, independent variates, with mean 0, ranging between $-\epsilon$ and $+\epsilon$.

Let $\phi_n(x\epsilon)$ be the probability that u_n lies between x and $x + \Delta x$.

Let $P_n(x\epsilon)$ be the probability that u_n lies between $-x$ and x. The mathematical expressions for $\epsilon\phi_n(x\epsilon)$ and $P_n(x\epsilon)$ are:

$$\epsilon\phi_n(x\epsilon) = \frac{1}{\prod_{k=0}^{n} \binom{n}{k}} \; \frac{1}{\pi} \int_0^{\infty} \frac{\cos\left(wx\right) \prod_{k=0}^{n} \sin\left[\binom{n}{k} w\right]}{w^{n+1}} \, dw$$

$$P_n(x\epsilon) = \int_{-x}^{x} \phi_n(y) \, dy$$

x	$\epsilon\phi_1(x\epsilon)$	$P_1(x\epsilon)$	x	$\epsilon\phi_2(x\epsilon)$	$P_2(x\epsilon)$	x	$\epsilon\phi_3(x\epsilon)$	$P_3(x\epsilon)$
0.0	0.50000	0.00000	0.0	0.25000	0.00000	0.0	0.14815	0.00000
0.2	0.45000	0.19000	0.2	0.24875	0.09983	0.2	0.14761	0.05919
0.4	0.40000	0.36000	0.4	0.24500	0.19867	0.4	0.14607	0.11796
0.6	0.35000	0.51000	0.6	0.23875	0.29550	0.6	0.14365	0.17593
0.8	0.30000	0.64000	0.8	0.23000	0.38933	0.8	0.14044	0.23277
1.0	0.25000	0.75000	1.0	0.21875	0.47917	1.0	0.13657	0.28819
1.2	0.20000	0.84000	1.2	0.20500	0.56400	1.2	0.13215	0.34196
1.4	0.15000	0.91000	1.4	0.18875	0.64283	1.4	0.12728	0.39385
1.6	0.10000	0.96000	1.6	0.17000	0.71467	1.6	0.12207	0.44373
1.8	0.05000	0.99000	1.8	0.14875	0.77850	1.8	0.11665	0.49148
2.0	0.00000	1.00000	2.0	0.12500	0.83333	2.0	0.11111	0.53704
			2.2	0.10125	0.87850	2.2	0.10556	0.58037
			2.4	0.08000	0.91467	2.4	0.10000	0.62148
			2.6	0.06125	0.94283	2.6	0.09444	0.66037
			2.8	0.04500	0.96400	2.8	0.08889	0.69704
			3.0	0.03125	0.97917	3.0	0.08333	0.73148
			3.2	0.02000	0.98933	3.2	0.07778	0.76370
			3.4	0.01125	0.99550	3.4	0.07222	0.79370
			3.6	0.00500	0.99867	3.6	0.06667	0.82148
			3.8	0.00125	0.99983	3.8	0.06111	0.84704
			4.0	0.00000	1.00000	4.0	0.05556	0.87037
						4.2	0.05001	0.89148
						4.4	0.04452	0.91039
						4.6	0.03914	0.92711
						4.8	0.03393	0.94172
						5.0	0.02894	0.95428
						5.2	0.02422	0.96490
						5.4	0.01984	0.97370
						5.6	0.01585	0.98083
						5.8	0.01231	0.98645
						6.0	0.00926	0.99074
						6.2	0.00675	0.99393
						6.4	0.00474	0.99621
						6.6	0.00318	0.99778
						6.8	0.00200	0.99880
						7.0	0.00116	0.99942
						7.2	0.00059	0.99976
						7.4	0.00025	0.99993
						7.6	0.00007	0.99999
						7.8	0.00001	1.00000
						8.0	0.00000	1.00000

x	$\epsilon\phi_4(x\epsilon)$	$P_4(x\epsilon)$
0.0	0.07726	0.00000
0.2	0.07720	0.03090
0.4	0.07705	0.06175
0.6	0.07679	0.09252
0.8	0.07643	0.12317
1.0	0.07597	0.15365
1.2	0.07541	0.18393
1.4	0.07476	0.21397
1.6	0.07401	0.24372
1.8	0.07318	0.27317
2.0	0.07227	0.30226
2.2	0.07127	0.33097
2.4	0.07021	0.35927
2.6	0.06907	0.38712
2.8	0.06787	0.41451
3.0	0.06661	0.44141
3.2	0.06529	0.46779
3.4	0.06391	0.49364
3.6	0.06248	0.51892
3.8	0.06100	0.54361
4.0	0.05946	0.56771
4.2	0.05787	0.59118
4.4	0.05623	0.61400
4.6	0.05454	0.63616
4.8	0.05281	0.65763
5.0	0.05103	0.67840
5.2	0.04921	0.69844
5.4	0.04735	0.71776
5.6	0.04547	0.73632
5.8	0.04357	0.75413
6.0	0.04167	0.77118
6.2	0.03976	0.78747
6.4	0.03786	0.80299
6.6	0.03598	0.81776
6.8	0.03413	0.83178
7.0	0.03231	0.84506
7.2	0.03053	0.85763
7.4	0.02879	0.86949
7.6	0.02710	0.88067
7.8	0.02546	0.89118
8.0	0.02387	0.90104
8.2	0.02234	0.91028
8.4	0.02085	0.91892
8.6	0.01942	0.92697
8.8	0.01804	0.93446

x	$\epsilon\phi_4(x\epsilon)$	$P_4(x\epsilon)$
9.0	0.01671	0.94141
9.5	0.01362	0.95654
10.0	0.01085	0.96875
10.5	0.00841	0.97835
11.0	0.00629	0.98568
11.5	0.00450	0.99105
12.0	0.00304	0.99479
12.5	0.00190	0.99723
13.0	0.00107	0.99869
13.5	0.00053	0.99947
14.0	0.00022	0.99983
14.5	0.00007	0.99996
15.0	0.00001	0.99999
15.5	0.00000	1.00000
16.0	0.00000	1.00000

x	$\epsilon\phi_5(x\epsilon)$	$P_5(x\epsilon)$
0.0	0.04151	0.00000
0.5	0.04145	0.04149
1.0	0.04128	0.08286
1.5	0.04099	0.12400
2.0	0.04060	0.16480
2.5	0.04011	0.20517
3.0	0.03953	0.24499
3.5	0.03885	0.28419
4.0	0.03810	0.32267
4.5	0.03727	0.36036
5.0	0.03638	0.39719
5.5	0.03542	0.43309
6.0	0.03440	0.46800
6.5	0.03333	0.50188
7.0	0.03223	0.53466
7.5	0.03108	0.56631
8.0	0.02990	0.59680
8.5	0.02870	0.62610
9.0	0.02747	0.65419
9.5	0.02624	0.68105
10.0	0.02500	0.70667
10.5	0.02375	0.73105
11.0	0.02251	0.75418
11.5	0.02128	0.77607
12.0	0.02005	0.79674
12.5	0.01884	0.81618
13.0	0.01764	0.83441
13.5	0.01646	0.85146
14.0	0.01530	0.86734
14.5	0.01417	0.88207
15.0	0.01306	0.89568
15.5	0.01199	0.90820
16.0	0.01095	0.91967
16.5	0.00995	0.93011
17.0	0.00899	0.93958
18.0	0.00720	0.95574
19.0	0.00561	0.96851
20.0	0.00425	0.97833
21.0	0.00311	0.98565
22.0	0.00220	0.99093
23.0	0.00149	0.99459
24.0	0.00095	0.99700
25.0	0.00056	0.99849
26.0	0.00030	0.99933
27.0	0.00014	0.99975
28.0	0.00005	0.99993
29.0	0.00001	0.99999
30.0	0.00000	1.00000
31.0	0.00000	1.00000
32.0	0.00000	1.00000

x	$\epsilon\phi_6(x\epsilon)$	$P_6(x\epsilon)$	x	$\epsilon\phi_6(x\epsilon)$	$P_6(x\epsilon)$	x	$\epsilon\phi_7(x\epsilon)$	$P_7(x\epsilon)$
0.0	0.02154	0.00000	25.0	0.00898	0.83832	0.0	0.01128	0.00000
0.5	0.02153	0.02154	25.5	0.00865	0.84713	0.5	0.01128	0.01128
1.0	0.02151	0.04306	26.0	0.00832	0.85562	1.0	0.01127	0.02255
1.5	0.02148	0.06455	26.5	0.00800	0.86377	1.5	0.01127	0.03382
2.0	0.02143	0.08600	27.0	0.00768	0.87161	2.0	0.01126	0.04509
2.5	0.02137	0.10740	27.5	0.00737	0.87913	2.5	0.01125	0.05634
3.0	0.02129	0.12874	28.0	0.00706	0.88634	3.0	0.01124	0.06759
3.5	0.02121	0.14999	28.5	0.00676	0.89325	3.5	0.01122	0.07882
4.0	0.02110	0.17114	29.0	0.00647	0.89986	4.0	0.01121	0.09004
4.5	0.02099	0.19219	29.5	0.00618	0.90619	4.5	0.01119	0.10124
5.0	0.02086	0.21312	30.0	0.00590	0.91222	5.0	0.01117	0.11242
5.5	0.02072	0.23391	30.5	0.00562	0.91798	5.5	0.01115	0.12358
6.0	0.02057	0.25456	31.0	0.00536	0.92347	6.0	0.01112	0.13471
6.5	0.02041	0.27505	31.5	0.00510	0.92870	6.5	0.01110	0.14582
7.0	0.02023	0.29537	32.0	0.00484	0.93367	7.0	0.01107	0.15691
7.5	0.02004	0.31551	33.0	0.00436	0.94286	7.5	0.01104	0.16796
8.0	0.01984	0.33545	34.0	0.00390	0.95111	8.0	0.01101	0.17898
8.5	0.01963	0.35519	35.0	0.00347	0.95847	8.5	0.01097	0.18997
9.0	0.01941	0.37471	36.0	0.00306	0.96500	9.0	0.01094	0.20093
9.5	0.01918	0.39401	37.0	0.00269	0.97075	9.5	0.01090	0.21185
10.0	0.01893	0.41306	38.0	0.00234	0.97578	10.0	0.01086	0.22272
10.5	0.01868	0.43186	39.0	0.00202	0.98014	10.5	0.01082	0.23356
11.0	0.01841	0.45041	40.0	0.00173	0.98389	11.0	0.01077	0.24435
11.5	0.01814	0.46869	41.0	0.00147	0.98708	11.5	0.01073	0.25510
12.0	0.01786	0.48669	42.0	0.00123	0.98977	12.0	0.01068	0.26581
12.5	0.01757	0.50440	43.0	0.00102	0.99201	12.5	0.01063	0.27646
13.0	0.01727	0.52181	44.0	0.00083	0.99386	13.0	0.01058	0.28707
13.5	0.01696	0.53893	45.0	0.00067	0.99536	13.5	0.01053	0.29762
14.0	0.01665	0.55573	46.0	0.00053	0.99655	14.0	0.01047	0.30812
14.5	0.01632	0.57222	47.0	0.00041	0.99749	14.5	0.01042	0.31857
15.0	0.01600	0.58838	48.0	0.00031	0.99821	15.0	0.01036	0.32896
15.5	0.01566	0.60421	49.0	0.00023	0.99876	15.5	0.01030	0.33929
16.0	0.01533	0.61971	50.0	0.00017	0.99916	16.0	0.01024	0.34957
16.5	0.01498	0.63486	51.0	0.00012	0.99945	16.5	0.01018	0.35978
17.0	0.01464	0.64967	52.0	0.00008	0.99966	17.0	0.01012	0.36993
17.5	0.01429	0.66414	53.0	0.00006	0.99979	17.5	0.01005	0.38001
18.0	0.01393	0.67825	54.0	0.00003	0.99988	18.0	0.00999	0.39003
18.5	0.01358	0.69200	55.0	0.00002	0.99994	18.5	0.00992	0.39998
19.0	0.01322	0.70540	56.0	0.00001	0.99997	19.0	0.00985	0.40987
19.5	0.01286	0.71844	57.0	0.00001	0.99999	19.5	0.00978	0.41968
20.0	0.01250	0.73112	58.0	0.00000	1.00000	20.0	0.00971	0.42943
20.5	0.01214	0.74344	59.0	0.00000	1.00000	20.5	0.00964	0.43910
21.0	0.01178	0.75540	60.0	0.00000	1.00000	21.0	0.00956	0.44870
21.5	0.01142	0.76700	61.0	0.00000	1.00000	21.5	0.00949	0.45822
22.0	0.01106	0.77824	62.0	0.00000	1.00000	22.0	0.00941	0.46767
22.5	0.01071	0.78913	64.0	0.00000	1.00000	22.5	0.00933	0.47705
23.0	0.01036	0.79966				23.0	0.00925	0.48634
23.5	0.01001	0.80984				23.5	0.00917	0.49555
24.0	0.00966	0.81968				24.0	0.00909	0.50469
24.5	0.00932	0.82917				24.5	0.00901	0.51374

x	$\epsilon\phi_7(x\epsilon)$	$P_7(x\epsilon)$	x	$\epsilon\phi_7(x\epsilon)$	$P_7(x\epsilon)$	x	$\epsilon\phi_7(x\epsilon)$	$P_7(x\epsilon)$
25.0	0.00893	0.52271	50.0	0.00433	0.85478	90	0.00025	0.99678
25.5	0.00885	0.53160	50.5	0.00424	0.85907	91	0.00022	0.99725
26.0	0.00876	0.54040	51.0	0.00416	0.86327	92	0.00019	0.99766
26.5	0.00868	0.54912	51.5	0.00407	0.86738	93	0.00017	0.99802
27.0	0.00859	0.55775	52.0	0.00399	0.87141	94	0.00015	0.99834
27.5	0.00850	0.56630	52.5	0.00390	0.87535	95	0.00013	0.99861
28.0	0.00841	0.57476	53.0	0.00382	0.87921	96	0.00011	0.99885
28.5	0.00833	0.58313	53.5	0.00374	0.88299	97	0.00009	0.99905
29.0	0.00824	0.59141	54.0	0.00365	0.88669	98	0.00008	0.99923
29.5	0.00815	0.59960	54.5	0.00357	0.89030	99	0.00007	0.99937
30.0	0.00806	0.60770	55.0	0.00349	0.89383	100	0.00006	0.99950
30.5	0.00796	0.61571	55.5	0.00341	0.89729	101	0.00005	0.99960
31.0	0.00787	0.62363	56.0	0.00333	0.90066	102	0.00004	0.99968
31.5	0.00778	0.63146	56.5	0.00326	0.90396	103	0.00003	0.99975
32.0	0.00769	0.63919	57.0	0.00318	0.90717	104	0.00003	0.99981
32.5	0.00759	0.64683	57.5	0.00310	0.91032	105	0.00002	0.99986
33.0	0.00750	0.65438	58.0	0.00303	0.91338	106	0.00002	0.99989
33.5	0.00741	0.66183	58.5	0.00295	0.91637	107	0.00001	0.99992
34.0	0.00731	0.66919	59.0	0.00288	0.91929	108	0.00001	0.99994
34.5	0.00722	0.67646	59.5	0.00281	0.92213	109	0.00001	0.99996
35.0	0.00712	0.68363	60.0	0.00273	0.92490	110	0.00001	0.99997
35.5	0.00703	0.69070	61.0	0.00259	0.93023	111	0.00000	0.99998
36.0	0.00693	0.69768	62.0	0.00246	0.93527	112	0.00000	0.99999
36.5	0.00684	0.70457	63.0	0.00232	0.94005	113	0.00000	0.99999
37.0	0.00674	0.71136	64.0	0.00219	0.94457	114	0.00000	0.99999
37.5	0.00665	0.71805	65.0	0.00207	0.94883	115	0.00000	1.00000
38.0	0.00655	0.72465	66.0	0.00195	0.95284	116	0.00000	1.00000
38.5	0.00645	0.73115	67.0	0.00183	0.95662	117	0.00000	1.00000
39.0	0.00636	0.73756	68.0	0.00172	0.96016	118	0.00000	1.00000
39.5	0.00626	0.74387	69.0	0.00161	0.96349	119	0.00000	1.00000
40.0	0.00617	0.75008	70.0	0.00150	0.96660	128	0.00000	1.00000
40.5	0.00607	0.75621	71.0	0.00140	0.96951			
41.0	0.00598	0.76223	72.0	0.00131	0.97222			
41.5	0.00588	0.76816	73.0	0.00122	0.97474			
42.0	0.00579	0.77400	74.0	0.00113	0.97709			
42.5	0.00569	0.77974	75.0	0.00105	0.97927			
43.0	0.00560	0.78539	76.0	0.00097	0.98128			
43.5	0.00551	0.79094	77.0	0.00089	0.98314			
44.0	0.00541	0.79640	78.0	0.00082	0.98486			
44.5	0.00532	0.80177	79.0	0.00076	0.98644			
45.0	0.00523	0.80704	80.0	0.00069	0.98789			
45.5	0.00514	0.81222	81.0	0.00063	0.98922			
46.0	0.00504	0.81731	82.0	0.00058	0.99043			
46.5	0.00495	0.82231	83.0	0.00053	0.99153			
47.0	0.00486	0.82722	84.0	0.00048	0.99254			
47.5	0.00477	0.83204	85.0	0.00043	0.99344			
48.0	0.00468	0.83676	86.0	0.00039	0.99426			
48.5	0.00459	0.84140	87.0	0.00035	0.99500			
49.0	0.00450	0.84595	88.0	0.00031	0.99566			
49.5	0.00442	0.85041	89.0	0.00028	0.99626			

x	$\epsilon\phi_8(x\epsilon)$	$P_8(x\epsilon)$	x	$\epsilon\phi_8(x\epsilon)$	$P_8(x\epsilon)$	x	$\epsilon\phi_8(x\epsilon)$	$P_8(x\epsilon)$
0.0	0.00582	0.00000	25.0	0.00548	0.28523	50.0	0.00456	0.53825
0.5	0.00582	0.00582	25.5	0.00547	0.29071	50.5	0.00453	0.54279
1.0	0.00582	0.01163	26.0	0.00545	0.29617	51.0	0.00451	0.54732
1.5	0.00582	0.01745	26.5	0.00544	0.30162	51.5	0.00449	0.55182
2.0	0.00582	0.02327	27.0	0.00543	0.30705	52.0	0.00447	0.55629
2.5	0.00581	0.02908	27.5	0.00541	0.31247	52.5	0.00444	0.56075
3.0	0.00581	0.03490	28.0	0.00540	0.31788	53.0	0.00442	0.56518
3.5	0.00581	0.04071	28.5	0.00538	0.32327	53.5	0.00439	0.56958
4.0	0.00581	0.04652	29.0	0.00537	0.32864	54.0	0.00437	0.57397
4.5	0.00581	0.05232	29.5	0.00535	0.33400	54.5	0.00435	0.57832
5.0	0.00580	0.05813	30.0	0.00534	0.33935	55.0	0.00432	0.58266
5.5	0.00580	0.06393	30.5	0.00532	0.34468	55.5	0.00430	0.58697
6.0	0.00580	0.06973	31.0	0.00531	0.34999	56.0	0.00427	0.59126
6.5	0.00579	0.07553	31.5	0.00529	0.35529	56.5	0.00425	0.59552
7.0	0.00579	0.08132	32.0	0.00527	0.36057	57.0	0.00423	0.59976
7.5	0.00579	0.08711	32.5	0.00526	0.36584	57.5	0.00420	0.60397
8.0	0.00578	0.09289	33.0	0.00524	0.37109	58.0	0.00418	0.60816
8.5	0.00578	0.09867	33.5	0.00522	0.37632	58.5	0.00415	0.61232
9.0	0.00577	0.10445	34.0	0.00521	0.38154	59.0	0.00413	0.61646
9.5	0.00577	0.11022	34.5	0.00519	0.38673	59.5	0.00410	0.62058
10.0	0.00576	0.11599	35.0	0.00517	0.39192	60.0	0.00408	0.62467
10.5	0.00576	0.12175	35.5	0.00515	0.39708	60.5	0.00405	0.62873
11.0	0.00575	0.12750	36.0	0.00514	0.40222	61.0	0.00403	0.63277
11.5	0.00575	0.13325	36.5	0.00512	0.40735	61.5	0.00400	0.63678
12.0	0.00574	0.13899	37.0	0.00510	0.41246	62.0	0.00398	0.64077
12.5	0.00573	0.14473	37.5	0.00508	0.41755	62.5	0.00395	0.64474
13.0	0.00573	0.15046	38.0	0.00506	0.42262	63.0	0.00393	0.64868
13.5	0.00572	0.15618	38.5	0.00504	0.42767	63.5	0.00390	0.65259
14.0	0.00571	0.16189	39.0	0.00502	0.43271	64.0	0.00387	0.65648
14.5	0.00570	0.16760	39.5	0.00500	0.43772	64.5	0.00385	0.66034
15.0	0.00570	0.17330	40.0	0.00498	0.44271	65.0	0.00382	0.66417
15.5	0.00569	0.17899	40.5	0.00496	0.44769	65.5	0.00380	0.66798
16.0	0.00568	0.18467	41.0	0.00495	0.45264	66.0	0.00377	0.67177
16.5	0.00567	0.19035	41.5	0.00492	0.45758	66.5	0.00375	0.67553
17.0	0.00566	0.19601	42.0	0.00490	0.46249	67.0	0.00372	0.67926
17.5	0.00565	0.20167	42.5	0.00488	0.46739	67.5	0.00369	0.68297
18.0	0.00564	0.20731	43.0	0.00486	0.47226	68.0	0.00367	0.68665
18.5	0.00563	0.21295	43.5	0.00484	0.47712	68.5	0.00364	0.69031
19.0	0.00562	0.21858	44.0	0.00482	0.48195	69.0	0.00362	0.69394
19.5	0.00561	0.22419	44.5	0.00480	0.48676	69.5	0.00359	0.69754
20.0	0.00560	0.22980	45.0	0.00478	0.49155	70.0	0.00357	0.70112
20.5	0.00559	0.23540	45.5	0.00476	0.49632	70.5	0.00354	0.70467
21.0	0.00558	0.24098	46.0	0.00474	0.50106	71.0	0.00351	0.70820
21.5	0.00557	0.24655	46.5	0.00471	0.50579	71.5	0.00349	0.71170
22.0	0.00556	0.25212	47.0	0.00469	0.51049	72.0	0.00346	0.71517
22.5	0.00554	0.25767	47.5	0.00467	0.51517	72.5	0.00344	0.71862
23.0	0.00553	0.26320	48.0	0.00465	0.51983	73.0	0.00341	0.72204
23.5	0.00552	0.26873	48.5	0.00463	0.52447	73.5	0.00338	0.72544
24.0	0.00551	0.27424	49.0	0.00460	0.52909	74.0	0.00336	0.72881
24.5	0.00549	0.27974	49.5	0.00458	0.53368	74.5	0.00333	0.73215

x	$\epsilon\phi_8(x\epsilon)$	$P_8(x\epsilon)$	x	$\epsilon\phi_8(x\epsilon)$	$P_8(x\epsilon)$	x	$\epsilon\phi_8(x\epsilon)$	$P_8(x\epsilon)$
75.0	0.00330	0.73547	100	0.00207	0.86904	160	0.00029	0.99070
75.5	0.00328	0.73876	101	0.00202	0.87313	162	0.00026	0.99179
76.0	0.00325	0.74203	102	0.00198	0.87712	164	0.00023	0.99278
76.5	0.00323	0.74527	103	0.00193	0.88103	166	0.00021	0.99367
77.0	0.00320	0.74848	104	0.00189	0.88485	168	0.00019	0.99447
77.5	0.00318	0.75167	105	0.00184	0.88858	170	0.00017	0.99519
78.0	0.00315	0.75483	106	0.00180	0.89223	172	0.00015	0.99583
78.5	0.00312	0.75797	107	0.00176	0.89579	174	0.00013	0.99641
79.0	0.00310	0.76108	108	0.00172	0.89926	176	0.00012	0.99691
79.5	0.00307	0.76416	109	0.00168	0.90266	178	0.00011	0.99736
80.0	0.00305	0.76722	110	0.00164	0.90597	180	0.00009	0.99775
80.5	0.00302	0.77026	111	0.00159	0.90920	182	0.00008	0.99810
81.0	0.00299	0.77326	112	0.00155	0.91235	184	0.00007	0.99840
81.5	0.00297	0.77624	113	0.00152	0.91542	186	0.00006	0.99866
82.0	0.00294	0.77920	114	0.00148	0.91841	188	0.00005	0.99889
82.5	0.00292	0.78213	115	0.00144	0.92133	190	0.00004	0.99908
83.0	0.00289	0.78503	116	0.00140	0.92416	192	0.00004	0.99925
83.5	0.00287	0.78791	117	0.00136	0.92693	194	0.00003	0.99939
84.0	0.00284	0.79077	118	0.00133	0.92962	196	0.00003	0.99950
84.5	0.00282	0.79359	119	0.00129	0.93224	198	0.00002	0.99960
85.0	0.00279	0.79640	120	0.00126	0.93478	200	0.00002	0.99969
85.5	0.00276	0.79917	121	0.00122	0.93726	202	0.00002	0.99975
86.0	0.00274	0.80193	122	0.00119	0.93966	204	0.00001	0.99981
86.5	0.00271	0.80465	123	0.00115	0.94200	206	0.00001	0.99985
87.0	0.00269	0.80736	124	0.00112	0.94427	208	0.00001	0.99989
87.5	0.00266	0.81003	125	0.00109	0.94648	210	0.00001	0.99992
88.0	0.00264	0.81268	126	0.00105	0.94862	212	0.00000	0.99994
88.5	0.00261	0.81531	127	0.00102	0.95069	214	0.00000	0.99996
89.0	0.00259	0.81791	128	0.00099	0.95271	216	0.00000	0.99997
89.5	0.00256	0.82049	129	0.00096	0.95466	218	0.00000	0.99998
90.0	0.00254	0.82304	130	0.00093	0.95655	220	0.00000	0.99999
90.5	0.00252	0.82557	131	0.00090	0.95838	222	0.00000	0.99999
91.0	0.00249	0.82807	132	0.00087	0.96016	224	0.00000	0.99999
91.5	0.00247	0.83055	133	0.00084	0.96188	226	0.00000	1.00000
92.0	0.00244	0.83301	134	0.00082	0.96354	228	0.00000	1.00000
92.5	0.00242	0.83544	135	0.00079	0.96514	256	0.00000	1.00000
93.0	0.00239	0.83784	136	0.00076	0.96670			
93.5	0.00237	0.84023	137	0.00074	0.96820			
94.0	0.00235	0.84258	138	0.00071	0.96965			
94.5	0.00232	0.84492	139	0.00069	0.97105			
95.0	0.00230	0.84723	140	0.00066	0.97239			
95.5	0.00227	0.84951	142	0.00062	0.97495			
96.0	0.00225	0.85178	144	0.00057	0.97732			
96.5	0.00223	0.85402	146	0.00053	0.97952			
97.0	0.00220	0.85623	148	0.00049	0.98155			
97.5	0.00218	0.85842	150	0.00045	0.98342			
98.0	0.00216	0.86059	152	0.00041	0.98515			
98.5	0.00213	0.86274	154	0.00038	0.98673			
99.0	0.00211	0.86486	156	0.00035	0.98817			
99.5	0.00209	0.86696	158	0.00032	0.98949			

References

[1] Cramér, Harald, Mathematical Methods of Statistics, Princeton University Press, (1946).

[2] Lowan, Arnold N., and Jack Laderman, 'On the distribution of errors in nth tabular differences,' Annals of Math. Statistics, Vol 10 (1939) 360.

ON A CONJECTURE OF KARL PEARSON

Henry E. Fettis

In 1906 K. Pearson (Reference [1]) wrote a memoir on the theory of random migrations, and some of his results have given rise to a number of contributions since then. Using, in part, Kluyver's investigations (Reference [2]), Pearson derived the probability for an individual of the population to be within a distance r from the center of dispersion, after n flights. The derivation is also given in Reference [3], section 13.48. The cumulative probability function for n flights of unit length is

$$P_n(r) = \int_0^r t\, F_n(t)\, dt \tag{1.0}$$

where

$$F_n(r) = \int_0^\infty y\, J_o(ry)\, [J_o(y)]^n\, dy \tag{1.1}$$

It can be readily shown that

$$P_n(r) = \int_0^\infty r\, J_1(ry)\, [J_o(y)]^n\, dy \tag{1.2}$$

where $J_k(y)$ is the Bessel function of the first kind of order k. Pearson tabulated $F_n(r)/2$ for $n \leq 7$, for r ranging between 0 and n (all that is necessary). He used a graphical procedure in getting his results, and remarked that for $n = 5$ the function appeared to be constant over the range between 0 and 1. He states: 'From $r = 0$ to $r = L$ (here 1) the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a *straight* line. . . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines.'

Greenwood and Duncan (Reference [4]) later extended Pearson's work for n=6(1)24, and more recently Scheid (Reference [5]) gave results for lower values of n (2 to 6) obtained by a Monte Carlo procedure. The function $F_5(r)$ was computed for $r < 1$ on the Remington-Rand 1103 computer. The results are given in Table 1, and although the function is not constant, it differs from 1/3 by less than .0034 in this range. This settles Pearson's conjecture. The table given on page 51 may help investigators of Monte Carlo techniques to compare their results with the known values.

Since the function $F_5(r)$ is so nearly constant in the range between 0 and 1, it would be reassuring if the question of constancy could also be settled theoretically. This in fact can be done, and two proofs will be given below. Before developing the proofs, it will be helpful to make some observations on the nature of $F_5(r)$. From (1.0), with $r = 1$, we obtain with the aid of the mean value theorem

$$P_n(1) = F_n(t_1) \int_0^1 t\, dt = \frac{1}{2} F_n(t_1) \ , \quad \text{where} \quad 0 < t_1 < 1 \qquad (1.21)$$

On the other hand, when $r = 1$, (1.2) can be evaluated in closed form, for all n, and we obtain

$$P_n(1) = 1/(n+1) \qquad (1.22)$$

Setting $n = 5$ in (1.21) and (1.22), we get

$$F_5(t_1) = 1/3 \qquad (1.23)$$

Since there exists a value t_1, $0 < t_1 < 1$, for which $F_5(t_1) = 1/3$, it follows that if $F_5(r)$ were constant in this range, that constant would have to be 1/3. We can now develop the proofs that $F_5(r)$ is not constant between 0 and 1.

First Proof: We shall study the nature of $F'_5(r) = dF_5(r)/dr$. Since F_5 is an even function of r, it is clear that $F'_5(0) = 0$. We shall now show that $F'_5(r)$

has a jump at $r = 1$, that both the left-hand derivative and right-hand derivative at $r = 1$ are well-defined, and that $F_5(r\text{-}0)$ is different from zero. This is sufficient to prove that $F_5(r)$ is not constant over the range in question. Let

$$F_5(r) = \int_0^N \phi \, dy + \int_N^\infty \phi \, dy \tag{1.30}$$

where

$$\phi(y) = yJ_o(ry) \; [J_o(y)]^5 \tag{1.31}$$

and N is at present an unspecified large number, which will later be permitted to go to infinity.

Thus, we have

$$\left[\frac{dF_5}{dr}\right]_{r=1-0} = U_1 + U_2 \tag{1.32}$$

where

$$U_1 = -\int_0^N y^5 \, J_1(y) \; [J_o(y)]^5 \, dy \tag{1.33}$$

$$U_2 = \left[\frac{\partial}{\partial r} \int_N^\infty \phi \, dy\right]_{r=1-0} \tag{1.34}$$

We may dispose of (1.33) by setting $r = 1$, integrating by parts, and then letting N go to infinity, which yields

$$\lim_{N\to\infty} U_1 = [y^2 \; J_o(y)^6/6]_0^\infty - \frac{1}{3} \int_0^\infty y \; [J_o(y)]^6 \, dy = -\frac{1}{3} F_5(1) \tag{1.35}$$

since the bracketed expression vanishes at both limits. In (1.34) it is permissible to use the leading term of the asymptotic representations of $J_o(sy)$, $s = r$ and 1, thus

$$yJ_o(ry)\, J_o^5(y) = \frac{1}{\sqrt{r}} \left(\frac{2}{\pi}\right)^3 \left\{ \frac{\cos\left(ry - \frac{\pi}{4}\right) \left[\cos\left(y - \frac{\pi}{4}\right)\right]^5}{y^2} \right\} + 0\left(\frac{1}{y^3}\right) \tag{1.36}$$

Now

$$\cos^5 x = \frac{10 \cos x + 5 \cos 3x + \cos 5x}{16} \tag{1.37}$$

Using $x = y - \frac{\pi}{4}$, and substituting (1.37) into (1.36), we get a set of terms of the form

$$\frac{1}{y^2}\, a_j \left[\cos\left\{(j+r)y - (j+1)\frac{\pi}{4}\right\} + \cos\left\{(j-r)y - (j-1)\frac{\pi}{4}\right\}\right] + 0\left(\frac{1}{y^3}\right) \tag{1.38}$$

$$(j = 1,\ 3,\ 5)$$

After differentiating the first factor (1.36) with respect to r (namely $1/\sqrt{r}$) and integrating, the resulting expression goes to zero with $1/N^2$. Similarly, for the terms involving r in the numerator of (1.38), the only term that can lead to a non-zero value, in the neighborhood of $r = 1$, is the term with $(j-r)$, $j = 1$, and we seek an expression for

$$\lim_{r \to 1} \left(\frac{2}{\pi}\right)^3 \frac{\partial}{\partial r} \int_N^\infty \frac{\cos (1-r)y}{y^2}\, dy \tag{1.39}$$

We get, upon differentiating,

$$\frac{\partial}{\partial r} \int_N^{\infty} \frac{\cos (1-r)y}{y^2} dy = \int_N^{\infty} \frac{\sin (1-r)y}{y} dy \tag{1.40}$$

Now as $r \to 1$ from the left, $(1-r)y$ is positive. Hence, if we write $(1-r)y = w$, the integral in (1.40) goes over into

$$\int_{N(1-r)}^{\infty} \frac{\sin w}{w} dw \tag{1.41}$$

Now letting $r \to 1$, (1.41) reduces to

$$\int_0^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2}$$

Hence, using (1.39) and (1.34)

$$U_2 = \frac{5}{4\pi^2} \tag{1.42}$$

It can be verified that if r approaches unity from the right (1.40) must be replaced by

$$-\int_N^{\infty} \frac{\sin (r-1)y}{y} dy$$

Thus, U_2 approaches $\frac{-5}{4\pi^2}$ from the right, and U_1 is unaltered. We therefore

get, finally

$$\left[\frac{dF_5}{dr}\right]_{r=1-} = \frac{5}{4\pi^2} - \frac{1}{3}F_5(1)$$

$$\left[\frac{dF_5}{dr}\right]_{r=1+} = \frac{-5}{4\pi^2} - \frac{1}{3}F_5(1) \tag{1.43}$$

Using the tabulated value of $F_5(1)$, namely .336828, we get

$$\left[\frac{dF_5}{dr}\right]_{r=1-} = .014376 \tag{1.44}$$

This checks very well with the value obtained by three-point numerical differentiation in Table 1, which yields .01436, with the last place uncertain. But since we wish to have a proof which does not depend on the tabular result, we can also proceed otherwise. It has already been observed that if $F_5(r)$ were constant between 0 and 1, that constant would have to be 1/3. Hence from (1.43), constancy of $F_5(r)$ in this range demands

$$\left[\frac{dF_5}{dr}\right]_{r=1-} \equiv \frac{5}{4\pi^2} - \frac{1}{3}F_5(1) \equiv \frac{5}{4\pi^2} - \frac{1}{9} = 0$$

But $\left(\frac{5}{4\pi^2}\right) \neq \frac{1}{9}$. This contradicts the hypothesis that $F_5(r)$ is constant.

Second Proof:* In the range $0 \leq r \leq 1$, we may write

$$F_5(r) = \Sigma a_n J_o(\alpha_n r) \tag{2.00}$$

where α_n is the n^{th} zero of $J_o(x)$.

* This proof is due to Dr. Blanch.

It is well known that

$$a_n = \frac{2}{[J_1(\alpha_n)]^2} \int_0^1 r\, J_o(\alpha_n r)\, F_5(r)\, dr \tag{2.01}$$

Hence

$$a_n = \frac{2}{[J_1(\alpha_n)]^2} \int_0^\infty y\, [J_o(y)]^5\, dy \int_0^1 r\, J_o(\alpha_n r)\, J_o(yr)\, dr$$

Since

$$\int_0^1 r\, J_o(ry)\, J_o(\alpha_n r)\, dr = \frac{\alpha_n\, J_1(\alpha_n)\, J_o(y)}{\alpha_n^2 - y^2}$$

we get

$$a_n = \frac{2\,\alpha_n}{J_1(\alpha_n)}\, z_n \tag{2.02}$$

where

$$z_n = \int_0^\infty \frac{y[J_o(y)]^6}{\alpha_n^2 - y^2}\, dy \tag{2.03}$$

On the other hand, if $F_5(r)$ were constant, (2.01) would yield

$$a_n = \frac{2F_5}{\alpha_n\, J_1(\alpha_n)} \tag{2.04}$$

Comparing (2.04) with (2.02), we note that F_5 can be constant only if

$$z_n = \frac{F_5}{\alpha_n^2}$$

for all n. Thus we need

$$\int_0^\infty \frac{\alpha_m^2 y [J_o(y)]^6}{\alpha_m^2 - y^2} dy = \int_0^\infty \frac{\alpha_n^2 y [J_o(y)]^6}{\alpha_n^2 - y^2} dy \qquad (2.05)$$

for all m, n. Equation (2.05) is equivalent to

$$\int_0^\infty w\, dy = \int_0^\infty \frac{y^3 \; [J_o(y)]^6 \; dy}{(\alpha_m^2 - y^2)\; (\alpha_n^2 - y^2)} \equiv 0 \qquad (2.06)$$

for all m, n.

Note that the integrand in (2.06) is positive for $y < \alpha_m < \alpha_n$, negative for $\alpha_m < y < \alpha_n$, and positive for $y > \alpha_n$. Thus, it may be possible to prove that (2.06) is different from zero by carrying the integration in (2.06) to a relatively low upper limit A. This, in fact, is so. The evaluation was carried by straightforward quadrature up to 15 on the Burroughs E 101 computer with the first two zeros of $J_o(x)$ for α_m (= 2.4048255577) and α_n (= 5.5200781103), respectively. We get

$$\int_0^{15} w\, dy = - .0001634$$

For $y \geq 15$,

$$\frac{\alpha_m^{\,2}}{y^2} < .02571$$

$$\frac{\alpha_n^{\,2}}{y^2} < .13543$$

Hence

$$\frac{1}{(y^2 - \alpha_m^{\,2})\;(y^2 - \alpha_n^{\,2})} < \frac{1}{(.92429)\;(.86457)y^4} < \frac{1.1872}{y^6}$$

Also

$$|J_o(y)| < \sqrt{\frac{2}{\pi y}}\;\left(1 + \frac{1}{8y}\right) < \sqrt{\frac{2}{\pi y}}\;(1.00834)\;, \quad y > 15$$

Hence

$$0 \leq \frac{y^3\;[J_o(y)]^6}{(y^2 - \alpha_m^{\,2})\;(y^2 - \alpha_n^{\,2})} < \left(\frac{2}{\pi}\right)^3 \frac{(1.05084)\;(1.1872)}{y^4} < \frac{.32189}{y^4}$$

It follows that

$$0 < \int_{15}^{\infty} w\;dy \leq \frac{.1073}{(15)^3} < .0000318$$

and therefore

$$.0001634 < \int_0^{\infty} w\;dy - .0001316$$

which proves that $F_5(z)$ is not constant. Although this proof depends on numerical integration, it is an evaluation totally independent of the tabular entries given here; it is an integration over a relatively small range, and one that can be easily verified on a desk calculator. (But Pearson did not have at his disposal the excellent tables of Bessel functions which made the work on the Burroughs E 101 easy; nor did he have a high-speed computer which could generate the Bessel functions.)

Method of Computation of Entries in Table 1: The method consisted in straightforward quadrature of (1.1) from $y = 0$ to $y = 501.5$, and adding the 'tail' from 501.5 to ∞. This method was chosen, since the problem was used primarily as a means of testing several subroutines for computing trigonometric functions, Bessel functions, and long quadratures. Once the code was checked out, it was a matter of less than four hours of machine time to get all the results. The 'tail' integral from 501.5 to ∞ was computed on a desk calculator. It yielded the following values:

r	'tail'
0	.00001 532
.1	.00000 114
1.0	.00016 066

For other values of r in the range, the 'tail' was less than 76 units in the 8^{th} decimal place.

Although the complete tabulation of $F_5(r)$ and $P_5(r)$ would require values of the function up to $r = 5$, only values for $r \leq 1$ were computed, since this tabulation was enough to settle Pearson's conjecture. The values of $P_5(r)$ were obtained by integrating $tF_5(t)$ and the resulting $P_5(1)$ checks very well indeed with the theoretical value.

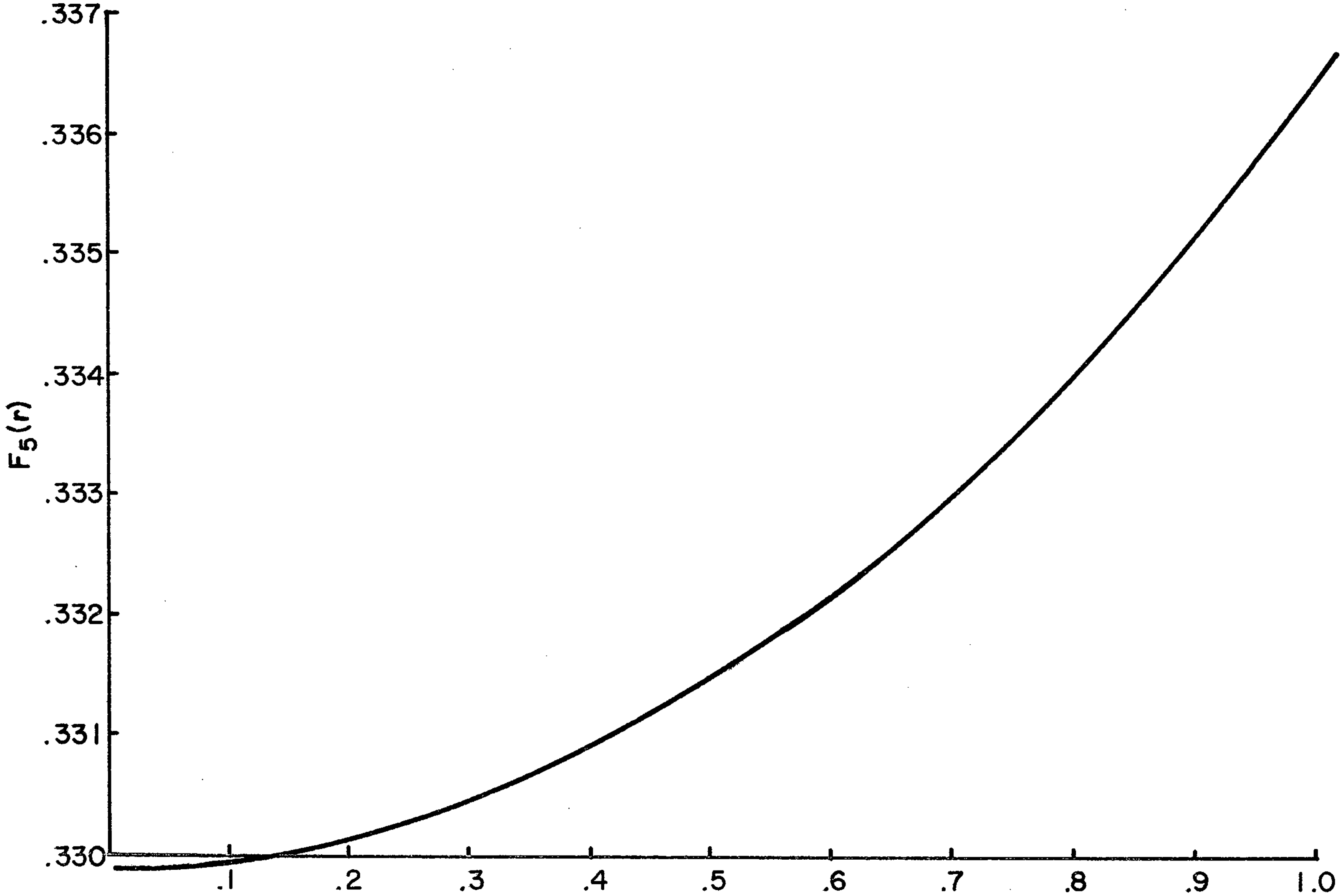
TABLE 1

r	$F_5(r)$	$\delta^2\ F_5$	$P_5(r)$	$\delta^2\ P_5$
0	.329933		.0	
.1	.330000	132	.001650	3302
.2	.330199	134	.006601	3308
.3	.330531	135	.014860	3318
.4	.330999	138	.026437	3332
.5	.331605	141	.041346	3350
.6	.332350	144	.059605	3373
.7	.333241	149	.081236	3400
.8	.334280	154	.106268	3432
.9	.335473	161	.134733	3470
1.0	.336828		.166667	

It may be worthwhile to compare some of Scheid's results, obtained by a Monte Carlo procedure with 6656 samples, with the entries obtainable from Table 1 by interpolation:

r	$P_5(r)$, by Table 1	Scheid's Value
.3125	.01613	.016
.9375	.14671	.146

It should also be noted that Pearson gave $F_5(r)/2\pi = .0537$. This yields $F_5(r) = .337$, which is slightly larger than the true value of $F_5(1)$.



Expanded Plot of $F_5(r)$ in the Interval $0 \leq r \leq 1$

The writer acknowledges the help of Dr. Gertrude Blanch in formulating the numerical calculations and the assistance of Mr. D. Clemm in the work on the desk calculators and in the calculation of certain economized expressions of Bessel functions.

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A PROBABILISTIC MODEL FOR RELIABILITY ESTIMATION FOR SPACE SYSTEM ANALYSIS

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and

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The remarks which follow describe one approach to the analysis and evaluation of complex space systems. These techniques have evolved in consonance with a philosophy which predicates that from the composite of successful experiences in the analysis and resolution of difficulties with specific systems, there can be synthesized a theoretically valid set of procedures which will be applicable and/or adaptable to other systems.

In this connection it is best, then, to orient our thinking to a specific class of missions in order that we may jointly agree on some of the difficulties associated with our task. Let us, for the sake of being specific, consider the problem of appraising the reliability of a scientific satellite designed for earth orbit. Thus, prior to actual orbit insertion, we are faced with the double-edged difficulty of insuring and demonstrating high system reliability while being denied the commonly-thought-of statistical tools with which to do this; namely, adequate quantities of test data on the overall system, obtained in its true operating environment. It also becomes apparent, when we speak of systems of the class just mentioned, that we have little involvement with mass-produced equipments; the systems of principal interest to us are complex, costly, and relatively few in number.

An important implication of this general picture is that although it may be impossible to establish directly the reliability of a system by testing it as a whole over the necessary time and in the required environment, we must nevertheless have some basis for ensuring that the maximum reliability attainable from the current state of the applicable arts has, in fact, been engineered into the systems. Since the ultimate purpose of a sensible reliability evaluation program is to provide this kind of assurance within the practical limitations of costs, time schedules, and the like, we are led to conclude that rational analysis providing quantitative knowledge about systems of interest must provide first for using the most *effective indirect techniques* for estimating (or predicting) the reliability of the system of interest.

It also becomes clear that such indirect techniques must ultimately consist of a functional analysis of the system, incorporating all the significant relevant information about the reliability of parts, components, and sub-systems either on the basis of tests in appropriate environments, wherever it is possible to make these, or on the basis of extrapolations to such environments, from the best data that are available.

If we adopt this point of view, and agree that for the class of systems considered here, only the indirect, analytical approach is truly feasible, then we can discuss a method for estimating the reliability of a complex system based on information obtained from environmental tests of sub-systems, components, and parts.

However, before plunging into the more substantive remarks pertaining to a probabilistic model for system evaluation, let us touch briefly on the subject of determining a rational set of sub-system goals which, when integrated into our probabilistic model, will produce at least a desired minimum goal for the system under consideration. Without going into the detailed considerations in this paper, let us simply assume as a first step that, as a result of an 'operational analysis approach' a minimum acceptable reliability-performance requirement, or goal, has been determined for the entire system and that this goal is consistent with the system's planned utilization. Having thus in hand a minimum reliability-performance goal, let us now state our definition of 'failure.' It is common practice to consider two types of failures, 'catastrophic,' and 'degradation' kinds. The first is usually associated with sudden, total failure of the system; the second with gradual deterioration where at some point in time system performance has drifted outside specified tolerance limits. For the sake of our discussion we will combine these two types of failure into one, and will simply state that the system fails at the time any of its essential physical characteristics attains values outside specified limits.

Having available these quantitative definitions of system performance goals and the associated definition of failure, one can then work backward to the primary sub-systems, and for each, one can unambiguously define quantitative reliability performance requirements and associated quantitative definitions of failure. In theory this process could be carried back to the parts level, but since we do not have parts available with both 'arbitrary' performance and reliability, it is readily discerned that within and between various sub-systems, 'trade-offs' to meet our quantitative performance and reliability goals will have to be made.

Without further belaboring the details of this procedure, let us state that there are two major benefits to be obtained by proceeding from the tolerance limits on each of the essential outputs of the overall system to the implied maximum allowable deviations in sub-system outputs. The two primary advantages of this approach from the analytical point of view are:

a. That one thereby obtains a valid quantitative definition of sub-system failures, and

b. In carrying out this process one has essentially derived a valid functional diagram of the system with the associated quantitative limits of required performance.

Once this background system analysis has been completed, one can look toward the evaluation procedure. Observing first, if for no more sophisticated reason than the observation that our satellite is expected to 'live' through the launch phase, and perform properly in the orbital phase, the following considerations must be incorporated in our probabilistic model. Not only must our probabilistic model be 'married' in a theoretically valid manner to our detailed functional diagram of the system, this model must also provide for the fact that our system is exposed to environments of differing degrees of severity over different intervals of time.

It may also be expected that a typical system will be designed to function in different modes over various intervals of time. All of these essential factors must be incorporated into our time-dependent probabilistic model. Further, where redundancy has been incorporated into the design, or where various functions are performed simultaneously, we must take into account the relevant series and parallel time-dependent arrangements in order that our model validly describes the system at hand.

Since it is not possible to treat in detail the many ramifications of an overall systems analysis in this paper, let us restrict our attention to the mathematical model for a 'typical' system. In order that this mathematical model will not be considered to be only an abstraction, let us consider that our system is comprised of the following sub-systems:

Booster

D. C. Power Supply

A. C. Power Supply

Telemetry

Attitude Control and Stabilization Systems

Satellite Ring Separation

Posigrade Rocket Firing

Satellite Tracking

We proceed by making the following definitions:

Overall System: All systems, sub-systems, components, and parts required for a satellite or spacecraft and booster for the defined mission from booster umbilical drop to the end of mission. This is usually specified in terms of a minimum time of successful operation of the satellite or accomplishment of a specific objective for a spacecraft.

System: A part of the overall system. The overall system is subdivided into systems, on the basis of functions, in order to facilitate determination of the reliability of the overall satellite system. Typical systems which would be considered to constitute an overall system have been noted above.

A mathematical model for the overall system may be defined by the following probabilistic relationship. After having expressed the overall system as systems, the probability of success of the overall system can then be given for a selected simple example by the following expression:

Pr(Overall System) = Pr(Booster) . Pr(D. C. Power | Booster) . Pr(Satellite Tracking | Booster, D. C. Power).

In this case the overall system consists of the booster, D.C. power, and tracking. Extension to any number of systems is obvious.

The expression $Pr(X)$ is the probability of success of system X and $Pr(Y|A, B, C, \ldots)$ is the conditional probability of success of system Y given the successful functioning of systems A, B, C, ... The purpose in expressing the overall system reliability as the product of conditional probabilities is to take into account possible dependencies among systems.

In detailing the mathematical model for a system, one must take into account the fact that a typical system may operate in different modes during different time periods. In such a case the total mission time for a system needs to be partitioned into k logical time periods as follows:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_k \qquad (1)$$

That is, the total mission time for the system is t_k; the i^{th} period is from t_{i-1} to t_i and of duration $\Delta t_i = t_i - t_{i-1}$. For example, if we consider the Attitude Control system during a typical normal mission, it operates in different

modes during the boost phase and the orbiting phase. In programs such as the Surveyor (a soft-landing lunar spacecraft) the landing phase may present a still different mode of operation for the Attitude Control system.

Let S_i represent the event that the system operates successfully to time t_i for $i = 0, 1, 2, \cdots, k$ and $P(S_i)$ the probability of event S_i. The quantity $P(S_i)$ is called the reliability of the system through time t_i and similarly $P(S_k)$ represents the reliability of the system for the mission of time t_k. Since successful operation of the system at time t_i implies successful operation from time t_0 to t_i it follows that:

$$S_i \subset S_{i-1} \qquad\qquad i = 1, 2, \cdots, k \tag{2}$$

and

$$S_i = S_1 \cap S_2 \cap \cdots \cap S_i \tag{3}$$

where S_i represents the set synonymous with the event S_i discussed above. Henceforth, standard set theory notation for inclusion ($\subset$), union ($\cup$), complement ($\tilde{\ }$), and intersection ($\cap$) will be used. Furthermore, from (3),

$$\begin{aligned} P(S_i) &= P(S_1 \cap S_2 \cap \cdots \cap S_i) \\ &= P(S_1).P(S_2|S_1) \cdots P(S_i|S_1 \cap S_2 \cap \cdots \cap S_{i-1}) \end{aligned} \tag{4}$$

where $P(A|B) = P(A \cap B) / P(B)$ and is the conditional probability of event A, given event B. Using (2), equation (4) can be expressed as:

$$P(S_i) = P(S_1) \cdot \frac{P(S_2)}{P(S_1)} \cdots \frac{P(S_i)}{P(S_{i-1})} \tag{5}$$

Thus, in order to find $P(S_i)$ in terms of the conditional probabilities, one must find $P(S_r)$ for $r = 1, 2, \cdots, i$.

It is considered desirable from an evaluation viewpoint to develop the overall mission reliability for the system in terms of the conditional probabilities shown in equation (5). In addition, one may observe that these latter conditional probabilities may be used to allocate the unreliability to the appropriate time periods. This can be shown as follows. Since by our definition the system either operates successfully or it fails, we can write that to time t_k:

$$G = S_k \cup \tilde{S}_k$$

where G represents the whole space, S_k the event of successful operation of the system from time 0 to time t_k, and $\tilde{S}_k$ the event of failure of the system during some time from 0 to t_k. Obviously

$$P(S_k) + P(\tilde{S}_k) = 1 \tag{6}$$

Moreover, it can be shown that

$$\tilde{S}_k = (S_0 \cap \tilde{S}_1) \cup (S_1 \cap \tilde{S}_2) \cup \cdots \cup (S_{k-1} \cap \tilde{S}_k) \tag{7}$$

where we define

$$S_0 = G$$

From (2), it follows that

$$\tilde{S}_{i-1} \subset \tilde{S}_i \tag{8}$$

In (7), the sets representing the intersections of S_{i-1} and $\tilde{S}_i (i = 1, 2, \cdots, k)$ are disjoint and hence

$$P(\tilde{S}_k) = \sum_{i=1}^{k} P(S_{i-1} \cap \tilde{S}_i) \tag{9}$$

Moreover, in view of (2) and (8), we can write

$$P(S_{i-1} \cap \tilde{S}_i) = P(S_{i-1}) - P(S_i)$$

and hence (9) becomes

$$P(\tilde{S}_k) = \sum_{i=1}^{k} \Delta_i P(S) \tag{10}$$

where $\Delta_i P(S) = P(S_{i-1}) - P(S_i)$ is the probability that the system functions up to time t_{i-1} and fails between t_{i-1} and t_i. In view of (5), the basic problem is to find the probabilities $P(S_i)$, $i = 1, 2, \cdots, k$; namely the reliability of the system up to time t_i. We do this next.

We now consider the development for obtaining the reliability up to a fixed time in the mission. We recall that a system is made up of sub-systems, components, and parts which have to function together in a certain manner in order for the systems to operate successfully. If a sub-system, component, or part fails, then the system may fail or not, depending on whether another sub-system, component, or part is available at the time of failure to perform the required function. The system is said to be successful during the i^{th} time period if there is at least one of each of the required sub-systems, components, and parts functioning in the intended manner during the i^{th} period. In view of the representation of sub-systems, components, and parts on so-called logic or network diagrams, one can define the group of sub-systems, components, and parts required for systems operation during the i^{th} period as a *path*. If all these work successfully it can be called a *successful path* for the i^{th} time period. It is therefore possible to define successful system operation to t_i, namely, the occurrence of the event S_i, by:

$$S_i = L_{i1} \cup L_{i2} \cup \cdots \cup L_{ir} \tag{11}$$

where L_{ij} represents a successful path from $t = 0$ to $t = t_i$, and the union of the events L_{ij} represents at least one successful path from 0 to t_i where there are r paths from time t_0 to time t_i. It is important to note that the events L_{ij} are not necessarily stochastically independent, since any two paths may have in them common components or parts. Furthermore, one can represent:

$$S_i = S_1^* \cap S_2^* \cap \cdots \cap S_i^* \tag{12}$$

and

$$S_r^* = P_{r1} \cup P_{r2} \cup \cdots \cup P_{r\ell_r}, \quad r = 1, 2, \cdots, i \tag{13}$$

where the term P_{rj} represents a successful path for the r^{th} time period only. Namely, P_{rj} represents the components for the j^{th} path which must function in order that the system functions from t_{r-1} to t_r and there are ℓ_r paths available during this time period. Hence, using (13) in (12), we have

$$S_i = \prod_{r=1}^{i} (P_{r1} \cup P_{r2} \cup \cdots \cup P_{r\ell_r}), \quad i = 1, \cdots, k \tag{14}$$

which represents the intersection of i sets S_r^*, namely, successful operation of at least one path during each time period up to time t_i.

Each path P_{rj} can in turn be represented as:

$$P_{rj} = A_{rj} \cap B_{rj} \cap \cdots \cap N_{rj} \qquad \begin{array}{l} r = 1, 2, \cdots, i \\ j = 1, 2, \cdots, \ell_r \end{array} \tag{15}$$

where the letters A_{rj}, B_{rj}, $\cdots$, N_{rj} represent the event of successful operation of sub-systems, components, or parts of the j^{th} path for the r^{th} time period.

The following definition is now made:

A sub-system, component, or part, operating as required during the r^{th} time period, is called *time-dependent* if

$$A_{rj} \subseteq A_{(r-1)j} \qquad r = 1, 2, \cdots, i \qquad j = 1, 2, \cdots, \ell_r \tag{16}$$

namely, the successful functioning of an equipment during the r^{th} period implies that it functioned successfully during all prior periods; namely, from time $t = 0$ to $t = t_r$. This definition is not dependent on the failure probability distribution. This distribution may perhaps be exponential, Weibull, or even discrete, such as the Binomial. In the former case it is assumed that the equipment must function from time 0 to time t_r; namely, the equipment must be 'hot' up to time t_r. For some 'discrete' situations, namely, parts, such as relays and switches, it may be assumed that once they function they have performed their work for the entire mission or if they fail, they have failed for the remainder of the mission. Hence, if (16) represents a go-no-go component:

$$A_{rj} = A_{(r-1)j} = A_j \text{ , say.} \tag{17}$$

Another way of stating this is that the event 'success of component A' is independent of the particular time period in which it is required to function.

It is obvious that events P_{rj} can be dependent or independent. For several systems which we have analyzed, it has been the case that the sub-systems, components, and parts of a system are all time-dependent, and also that $P(S_i)$ can be derived from the network paths of the i^{th} time period. If the systems consist of the same network paths during the $(i-1)^{st}$ and i^{th} time period, then in view of (15)

and (16)

$$P_{ij} \subset P_{(i-1)j} \qquad \begin{array}{l} i = 1, \cdots, k \\ j = 1, 2, \cdots, \ell_i \end{array} \tag{18}$$

and consequently

$$S_i^* \subset S_{i-1}^* \qquad i = 1, \cdots, k \tag{19}$$

From (12)

$$S_i = S_i^* \tag{20}$$

and

$$P(S_i) = P(S_i^*) \qquad i = 1, 2, \cdots, k$$

Hence, the functional block diagrams (that is, the logic network describing the paths) for the i^{th} time period provide all the information necessary to obtain the reliability up to time t_i. Furthermore, it can be shown that if:

$$P(S_{i-1}|P_{ij}) = 1 \qquad \begin{array}{l} i = 1, \cdots, k \\ j = 1, \cdots, \ell_i \end{array} \tag{21}$$

then again

$$P(S_i^*) = P(S_i)$$

It is noted that relationship (21) implies that if all the equipments of the j^{th} path are functioning successfully between time t_{i-1} and t_i, then the particular

system has functioned successfully up to time t_{i-1}. If this is true for all i and j, then again the equipment reliabilities and the network for the i^{th} period determine completely the system reliability up to the i^{th} period. This can be shown by the following argument.

We have shown

$$S_i = S_{i-1} \cap S_i^*$$

where

$$S_i^* = P_{i1} \cup P_{i2} \cup \cdots \cup P_{i\ell_i}$$

and

$$S_i = (S_{i-1} \cap P_{i1}) \cup (S_{i-1} \cap P_{i2}) \cup \cdots \cup (S_{i-1} \cap P_{i\ell_i}) \tag{22}$$

Hence, it follows that

$$P(S_i) = P\left(\bigcup_{j=1}^{\ell_i} S_{i-1} \cap P_{ij}\right)$$

and also

$$P(S_i) = \sum_{j=1}^{\ell_i} P(S_{i-1} \cap P_{ij}) - \sum_{j,j'} P(S_{i-1} \cap P_{ij} \cap P_{ij'})$$

$$+ \cdots \pm P(S_{i-1} \cap P_{i1} \cap \cdots \cap P_{i\ell_i})$$

$$= \sum_{j=1}^{\ell_i} P(S_{i-1}|P_{ij})\, P(P_{ij}) \qquad (23)$$

$$- \sum_{j,j'} P(S_{i-1}|P_{ij} \cap P_{ij'})\, P(P_{ij} \cap P_{ij'})$$

$$+ \cdots \pm P(S_{i-1}|P_{i1} \cap \cdots \cap P_{i\ell_j})\, P(P_{i1} \cap P_{i2} \cap \cdots \cap P_{i\ell_j})$$

It is further noted that the right-hand side of (23) reduces to

$$P(P_{i1} \cup P_{i2} \cup \cdots \cup P_{i\ell_j})$$

if all the conditional probabilities are taken as unity. Also, since we can write:

$$S_i = S_{i-1} \cap \bigcup_{j=1}^{\ell_i} P_{ij}$$

and

$$S_{i-1} \cap \bigcup_{j=1}^{a} P_{ij} \subset S_i \subset \bigcup_{j=1}^{\ell_i} P_{ij}, \qquad a < \ell_i$$

it follows that

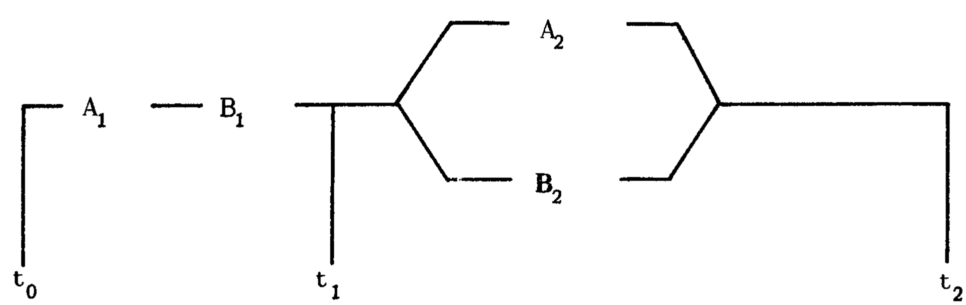
$$P\left(S_{i-1} \cap \bigcup_{j=1}^{a} P_{ij}\right) \leq P(S_i) \leq P\left(\bigcup_{j=1}^{\ell_i} P_{ij}\right) = P(S_i^*) \tag{24}$$

Equation (24) gives upper and lower bounds on the desired reliability of the systems up to time t_i.

We shall now describe these generalized ideas in a simplified example. We assume that an overall system consists of two independent systems, A and B, and that A and B operate in series during the first time period and in parallel during the second time period. Furthermore, we assume that both systems are time-dependent and that we wish to determine overall system reliability. We divide the total operating time into the two periods defined by

$$0 = t_0 < t_1 < t_2 \tag{25}$$

We can then describe the operation of the overall system by the following schematic:



The subscripts on A and B indicate operation of the respective systems during periods one and two. If we let S_i represent the event that the overall system is working satisfactorily up to time t_i, where $i = 1, 2,$ and $\tilde{S}_i$ the complement of this event, then we can consider the probabilities for all possibilities as follows:

t_1 to t_2 \ t_0 to t_1		State of overall system at t_1		
		S_1	$\tilde{S}_1$	Total
State of overall system at t_2	S_2	$P(S_1 \cap S_2)$	$P(\tilde{S}_1 \cap S_2)$	$P(S_2)$
	$\tilde{S}_2$	$P(S_1 \cap \tilde{S}_2)$	$P(\tilde{S}_1 \cap \tilde{S}_2)$	$1 - P(S_2)$
Total		$P(S_1)$	$1 - P(S_1)$	1

The marginal totals are obtained as follows:

$$P(S_1 \cap S_2) = P(S_2)$$

since

$$S_2 \subset S_1$$

Also,

$$P(\tilde{S}_1 \cap S_2) = 0$$

since overall system failure at t_1 precludes success at time t_2. Also

$$P(S_1 \cap \tilde{S}_2) = P(S_1) - P(S_2)$$

and since

$$\tilde{S}_1 \subset \tilde{S}_2$$

$$P(\tilde{S}_1 \cap \tilde{S}_2) = P(\tilde{S}_1) = 1 - P(S_1)$$

Furthermore, during the first time period we have only one successful path, namely

$$P_{11} = A_1 \cap B_1$$

It is noted that the same symbol, namely a capital letter, is used to identify a system, and also to indicate successful system operation of this system.

During the second time period, we have two successful paths

$$P_{21} = A_2$$

$$P_{22} = B_2$$

We assumed that the systems are time-dependent and hence

$$A_2 \subseteq A_1 \quad \text{and} \quad B_2 \subseteq B_1$$

Continuing to use the notation developed previously, we obtain

$$\begin{aligned} S_2 &= P_{11} \cap (P_{21} \cup P_{22}) \\ &= (A_1 \cap B_1) \cap (A_2 \cup B_2) \\ &= (A_1 \cap B_1 \cap A_2) \cup (A_1 \cap B_1 \cap B_2) \\ &= (A_2 \cap B_1) \cup (A_1 \cap B_2) \end{aligned}$$

It is obvious that S_2, as defined above, is a sub-set of S_1, namely of

$$A_1 \cap B_1$$

If we carry through a similar development for the other possible states, we obtain the following results for the four cells of the schematic:

$$P(S_1 \cap S_2) = P(B_1 \cap A_2) + P(B_2 \cap A_1) - P(A_2 \cap B_2)$$

$$P(S_1 \cap \tilde{S}_2) = [P(A_1) - P(A_2)]\ [P(B_1) - P(B_2)]$$

$$P(\tilde{S}_1 \cap \tilde{S}_2) = 1 - P(A_1 \cap B_1)$$

$$P(\tilde{S}_1 \cap S_2) = 0$$

and the sum of the four probabilities adds to 1, if we recall that A and B are independent.

For the marginal probabilities we obtain:

$$P(S_1) = P(A_1 \cap B_1)$$

$$P(\tilde{S}_1) = 1 - P(A_1 \cap B_1)$$

$$P(S_2) = P(S_1 \cap S_2)$$

as given above, and

$$P(\tilde{S}_2) = 1 - P(B_2 \cap A_1) - P(A_2 \cap B_1) + P(A_2 \cap B_2)$$

We can also obtain the probability of the event S_2 by using equation (23) developed previously, and obtain

$$P(S_2) = P(S_1|P_{21})\ P(P_{21}) + P(S_1|P_{22})\ P(P_{22}) - P(S_1|P_{21} \cap P_{22})\ P(P_{21} \cap P_{22})$$

and since

$$P(S_1\ P_{21}) = P(B_1)$$

$$P(S_1\ P_{22}) = P(A_1)$$

$$P(S_1\ P_{21} \cap P_{22}) = 1$$

we have

$$P(S_2) = P(B_1)\ P(A_2) + P(A_1)\ P(B_2) - P(A_2 \cap B_2)$$

which is the same result obtained above. In this example the systems A and B have, of course, been assumed to be independent. It is important to differentiate between the successful operation of the system and the successful operation of the overall system. In the above example, although A and B are assumed redundant during the second time period, A and B must both function during the first time period in order that this redundancy is in fact available during the second time interval.

It is possible that to one who has never attempted to estimate 'indirectly' the reliability of a complex system as the synthesis of the reliabilities of its constituent 'functional' sub-units, the foregoing remarks may appear to outline a rather foreboding problem. In this regard, it is regretted that time does not permit a detailed description in this paper of the adaptation of these procedures to large-scale computers for numerical solution. Let us simply state, that for the

logic networks a matrix algorism and symbolic expansion has been developed. The expansion is directly accomplished by a computer and provides the probabilistic equations for all possible paths for almost any series-parallel time-dependent functional configuration. In turn, another portion of the computer operation incorporates into this procedure the prior obtained individual estimates of the reliability for each of the constituent functions. The overall task to be associated with a large, complex, and highly redundant system is not simple. However, the procedures have been worked out and, as we have just indicated, they have also been effectively coded for a large-scale computer.

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EXACT CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR THE PARAMETER OF A ONE-PARAMETER NEGATIVE EXPONENTIAL POPULATION

H. Leon Harter

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Summary

A table is given of 6-decimal-place values of the coefficients of the m^{th} order statistic of a sample of size n from a one-parameter negative exponential population in exact upper and lower confidence bounds (confidence 1-P) for the parameter σ, for n = 1(1)20(2)40, P = .0001, .0005, .001, .005, .01, .025, .05, .1(.1).5. The interval between exact lower and upper confidence bounds, each associated with confidence 1-P, is, of course, an exact central confidence interval (confidence 1-2P). Expected values of the upper confidence bounds and expected lengths of the central confidence intervals are given to six decimal places, and the effectiveness of each, relative to the corresponding bound or interval based on the sample mean (the efficient estimator for σ), is given to the nearest tenth of one percent. The main table gives, for each combination of n and 1-P, results for that value of m which minimizes the expected value of the upper confidence bound. In most cases, the same value of m minimizes the expected length of the central confidence interval; for cases in which this is not true, the value of the expected length of the central confidence interval is marked with an asterisk. A second table gives, for these cases, results for that value of m which does minimize the expected length of the central confidence interval. A description is given of the method of computation of the tables. These tables are compared with others previously published, and a brief discussion of possible uses is given.

1. Introduction

During the past few years, several papers have been published on point and interval estimation of parameters of various populations by the use of order statistics or differences of order statistics (ranges and quasi-ranges). Harter (1959) gave theory and extensive tables for point estimation of the population standard deviation for a rectangular population, based on the sample range, and for a normal population, based on one quasi-range and on linear combinations of two quasi-ranges.

The estimator based on the sample range is the efficient estimator for a rectangular population, while in the case of a normal population the efficiencies of the estimators based on one quasi-range and on the best linear combination of two quasi-ranges are greater than or equal to the asymptotic values, approximately 65 percent and 80 percent, respectively. For asymmetric populations, however, the efficiency of estimators based on ranges and quasi-ranges is not so high. For example, in the case of a one-parameter negative exponential population, better estimates of the parameter, which is both mean and standard deviation, can be obtained from a single-order statistic than from a quasi-range. Harter (1961b) gave theory and tables for point estimation of the parameter for a one-parameter negative exponential population from one- or two-order statistics and of the parameters for a two-parameter negative exponential population from two-order statistics.

Interval estimation of population parameters from sample quasi-ranges has been considered in papers by Chu (1957) and by Chu, Leone, and Topp (1957). The method, outlined in the latter paper, of obtaining confidence bounds by first applying distribution-free methods and then imposing the distribution, was applied by Leone, Rutenberg, and Topp (1961), who gave tables of confidence intervals for the standard deviation of normal, one-parameter negative exponential, and rectangular populations. Harter (1961a) pointed out that the exact confidence intervals, in which the coefficients of the quasi-ranges are the reciprocals of percentage points of those quasi-ranges for the population under consideration, have expected length much shorter than the approximate confidence intervals given by Leone, Rutenberg, and Topp. He illustrated this point by giving tables of exact confidence intervals, based on the sample range, for the standard deviation of a rectangular population. A similar tabulation of exact confidence intervals for the standard deviation of a normal population, each based on one sample quasi-range, has been completed and will be submitted for publication in a technical journal.

Just as a single-order statistic will yield a more efficient point estimator of the parameter of a one-parameter negative exponential population than will a quasi-

range, it will also provide upper confidence bounds having smaller expected values and central confidence intervals having shorter expected length. The present paper gives tables of upper and lower confidence bounds (confidence 1-P) and central confidence intervals (confidence 1-2P) for the parameter of a one-parameter negative exponential population, based on the m^{th} order statistic of a sample of size n, for $P = .0001, .0005, .001, .005, .01, .025, .05, .1(.1).5$ and $n = 1(1)20(2)40$ for values of m which minimize the expected value of the upper confidence bound and/or the expected length of the central confidence interval.

2. Method of Computation of Tables

Consider the one-parameter negative exponential population, which has probability density function $f(x) = \sigma^{-1} e^{-x/\sigma}$, $0 \leq x < \infty$. If we set σ equal to 1 in order to standardize the population, we obtain $f(x) = e^{-x}$, from which we can find the cumulative distribution function $F(x) = \int_0^x e^{-x}\, dx = 1 - e^{-x}$. Wilks (1948) has shown that the probability element $\phi(x_m)dx_m$ for the m^{th} order statistic of a sample of size n from a population having probability density function $f(x)$ and cumulative distribution function $F(x)$ is given by

$$\phi(x_m)\, dx_m = \frac{\Gamma(n+1)}{\Gamma(m)\, \Gamma(n-m+1)} [F(x_m)]^{m-1} [1 - F(x_m)]^{n-m} f(x_m)\, dx_m \tag{1}$$

Substituting the values of $f(x)$ and $F(x)$ for the standardized one-parameter negative exponential population, and expanding, we obtain

$$\phi(x_m)\, dx_m = \sum_{k=0}^{m-1} \frac{(-1)^k\, \Gamma(n+1)}{\Gamma(n-m+1)\, \Gamma(m-k)\, \Gamma(k+1)}\, e^{-x_m(n-m+k+1)}\, dx_m \tag{2}$$

which may be rewritten in the form

$$\phi(x_m)\, dx_m = \sum_{k=1}^{m} \frac{(-1)^{k-1}\, \Gamma(n+1)}{\Gamma(n-m+1)\, \Gamma(m-k+1)\, \Gamma(k)}\, e^{-x_m(n-m+k)}\, dx_m \tag{3}$$

Integrating $\phi(x_m)\, dx_m$ between the limits 0 and x_m, we obtain the cumulative distribution function of x_m, which is given by

$$\Phi(x_m) = \sum_{k=1}^{m} (-1)^{k-1}\, C(n,m,k) - \sum_{k=1}^{m} (-1)^{k-1}\, C(n,m,k)\, e^{-x_m(n-m+k)} \tag{4}$$

where $C(n,m,k)$ is given by

$$C(n,m,k) = \frac{\Gamma(n+1)}{(n-m+k)\, \Gamma(n-m+1)\, \Gamma(m-k+1)\, \Gamma(k)} \tag{5}$$

It is obvious that as $x_m \to \infty$, $\Phi(x_m) \to 1$ and the second term on the right-hand side of (4) tends to zero; hence, the first term on the right-hand side of (4) must have the value 1, so that (4) may be rewritten in the form

$$\Phi(x_m) = 1 - \sum_{k=1}^{m} (-1)^{k-1}\, C(n,m,k)\, e^{-x_m(n-m+k)} \tag{6}$$

The first step in the computation of the tables was the tabulation, using double-precision arithmetic on the IBM 7090 computer, of the cumulative distribution function $\Phi(x_m)$, as defined by (6), for $x_m = 0(.05)\ 14.95$. For each value of n, this was done for several successive integer values of m centered about the

value which yields the most efficient point estimator of the population parameter [see Harter (1961b)]. It had been intended to perform these computations for n = 1(1)20(2)40(10)100, but loss of accuracy in summing the alternating series in (6), whose individual terms for large values of n may be very large even though the sum is small (≤ 1), dictated cutting off the computations at $n = 40$. Only single-precision (8-decimal-place) values of $\Phi(x_m)$ were printed out, but the double-precision values were retained in the memory for use in the next step.

The second step was inverse interpolation in the table of $\Phi(x_m)$, for each combination of n and m, to determine percentage points of x_m corresponding to cumulative probability P = .0001, .0005, .001, .005, .01, .025, .05, .1(.1).9, .95, .975, .99, .995, .999, .9995, .9999. Iterative linear inverse interpolation was performed, again using double-precision arithmetic on the IBM 7090 computer. This type of interpolation is accomplished as follows: Let $\Phi(x_{m1})$ and $\Phi(x_{m2})$ be two successive tabular values such that $\Phi(x_{m1}) < P < \Phi(x_{m2})$. Perform linear inverse interpolation to find a value x_{m0} between x_{m1} and x_{m2} corresponding to $\Phi(x_m) = P$. Compute $\Phi(x_{m0})$, using (6). If $\Phi(x_{m0}) < P$, replace x_{m1} by x_{m0} and $\Phi(x_{m0})$; if $\Phi(x_{m0}) > P$, replace x_{m2} by x_{m0} and $\Phi(x_{m2})$ by $\Phi(x_{m0})$. Then repeat the linear inverse interpolation. Continue until the relative difference between successive values of x_{m0} is less than 10^{-8}; then stop and set the percentage point equal to the last value of x_{m0}.

The third step was finding the reciprocals of the percentage points obtained in the second step. These reciprocals are the coefficients of x_m in exact upper confidence bounds associated with confidence 1-P (exact lower confidence bounds associated with confidence P) for the parameter σ. The interval between exact lower and upper confidence bounds, each associated with confidence 1-P, is an exact central confidence interval (confidence 1-2P).

The fourth step was finding the expected values of the confidence bounds and the expected lengths of the confidence intervals. The expected values of the

confidence bounds were found by multiplying the coefficients of x_m in these bounds, as determined in the third step, by the expected value of x_m, which is given [see Epstein and Sobel (1953)] by

$$E(x_m) = \sum_{j=1}^{m} \frac{1}{n - j + 1} \tag{7}$$

The expected lengths of the confidence intervals were then found by subtraction, since they are simply the differences of pairs of expected values of confidence bounds.

The fifth and final step in the computation of the tables was the determination of the effectiveness of the upper confidence bounds and of the central confidence intervals. Leone, Rutenberg, and Topp (1961) have defined the effectiveness of a substitute central confidence interval as the ratio, expressed as a percentage, of the expected length of the conventional confidence interval (in this case based on the sample mean) to that of the substitute interval. A logical extension results in the definition of the effectiveness of a substitute upper confidence bound (for a parameter which cannot be negative) as the ratio, expressed as a percentage, of the expected value of the conventional upper confidence bound to that of the substitute bound. (Since 0 is a lower confidence bound with confidence 1, the interval from 0 to the upper confidence bound with confidence 1-P is a non-central confidence interval with confidence 1-P.) For the case under consideration, that of a one-parameter negative exponential population, the conventional upper confidence bound (confidence 1-P), based on the sample mean, has expected value equal to $2n/\chi^2_{P,2n}$ and the conventional central confidence interval (confidence 1-2P), also based on the sample mean, has expected length equal to $2n(1/\chi^2_{P,2n} - 1/\chi^2_{1-P,2n})$. The first subscript on the chi-square represents the cumulative probability and the second is the number of degrees of freedom, n being the sample size. Values of the percentage points of

χ^2 have been tabulated by Hald and Sinkbaek (1950). Values are not given for P = .0001 and P = .9999, so the effectiveness has not been computed for 1-P = .9999 and 1-2P = .9998. Hald and Sinkbaek give only three significant figures for percentage points whose values are less than 1, which occur for P and f (the number of degrees of freedom) both small. Whenever possible in these cases, four significant figures were read from another table of percentage points or found by interpolation in a table of the probability integral of the χ^2 distribution, both of these latter tables given by Pearson and Hartley (1954). When the expected value (length) of the conventional confidence bound (interval) had been found, it was multiplied by 100 and the result was divided by the expected value (length) of the substitute bound (interval) to determine the percentage effectiveness of the substitute bound (interval).

For each sample size n, values of the coefficients of the m^{th} order statistic (for several values of m) in the exact upper and lower confidence bounds were rounded to 7 significant figures or 6 decimal places, whichever is less accurate, as were the expected values of the upper confidence bounds and the expected lengths of the central confidence intervals. The values of the effectiveness of the upper confidence bounds and that of the central confidence intervals were rounded to the nearest tenth of one percent. The results, together with the corresponding values of n, m, 1-P, and 1-2P, were punched on IBM cards. These cards were sorted manually. For each pair of values of n and 1-P, the card for that value of m which minimizes the expected value of the upper confidence bound (confidence 1-P) was placed in deck A. In most cases (all but 70 out of 330), the same value of m also minimizes the expected length of the central confidence interval (confidence 1-2P). For each of the 70 cases in which this is not true, an asterisk was punched after the expected length of the confidence interval in the card in deck A, and the card for the value of m which does minimize the expected length of the central confidence interval was placed in deck B. All cards not belonging either to deck A or to deck B were discarded. Table 1, with suitable title and with column headings

which are explained on the last page of the table, was reproduced from deck A, and Table 2, with suitable title and with the same column headings, was reproduced from deck B, using the IBM 407 tabulator in both cases.

3. Comparison with Other Tables

The effectiveness of the shortest exact central confidence intervals is in the neighborhood of 80 percent, except for very small samples, in which case it is even higher. This compares with an effectiveness in the neighborhood of 55 percent or even less for the approximate confidence intervals of Leone, Rutenberg, and Topp (1961). There is a two-fold reason for the difference, which is due partly to the superiority of the exact method over the approximation based on distribution-free methods and partly to the use of a single-order statistic instead of a quasi-range. The effectiveness of the exact central confidence intervals could have been made slightly higher by optimizing the upper and lower confidence bounds separately as Leone, Rutenberg, and Topp did for their approximate confidence intervals. It was decided that the small increase in effectiveness that would have resulted was not worth the additional complication of having the upper and lower confidence bounds based on different order statistics.

The effectiveness of the lowest exact upper confidence bounds is even higher than that of the shortest exact central confidence intervals. In this case, no other tables are available for purposes of comparison.

4. Possible Uses of the Tables

One advantage usually claimed for substitute estimates is that they are easier to compute than the conventional estimates. In this case, there is little advantage here, since the conventional estimates are based on the sample mean, which is itself relatively easy to compute. The principal uses of the present tables will probably be in obtaining interval estimates for the mean-time-to-failure of (say) electronic components on the basis of incomplete results of a life test, without waiting for all of the components in the sample placed on test to fail.

5. Remarks

Madansky (1962) has pointed out that minimizing the expected length of a confidence interval is not always a good criterion of the merit of the interval [see also Lehmann (1959)]. Further investigation is now under way to compare the results obtained by the use of this criterion with those obtained by the use of other criteria, and if possible to determine the best criterion. Particular attention is being given to the best choice of m in the problem discussed in this paper.

TABLE 1

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	9999.500	9999.500	0.108574	0.9998	9999.391			1	1
0.9995	1999.500	1999.500	0.131563	0.9990	1999.368	100.0	100.0	1	1
0.9990	999.4999	999.4999	0.144765	0.9980	999.3551	100.0	100.0	1	1
0.9950	199.4996	199.4996	0.188739	0.9900	199.3108	100.0	100.0	1	1
0.9900	99.49916	99.49916	0.217147	0.9800	99.28201	100.0	100.0	1	1
0.9750	39.49789	39.49789	0.271085	0.9500	39.22680	100.0	100.0	1	1
0.9500	19.49573	19.49573	0.333808	0.9000	19.16192	100.0	100.0	1	1
0.9000	9.491222	9.491222	0.434294	0.8000	9.056927	100.0	100.0	1	1
0.8000	4.481420	4.481420	0.621335	0.6000	3.860085	100.0	100.0	1	1
0.7000	2.803673	2.803673	0.830584	0.4000	1.973090	100.0	100.0	1	1
0.6000	1.957615	1.957615	1.091357	0.2000	0.866259	100.0	100.0	1	1
0.5000	1.442695	1.442695				100.0		1	1
0.9999	99.49916	149.2487	0.100975	0.9998	149.0973			2	2
0.9995	44.21947	66.32921	0.120570	0.9990	66.14836	94.4	94.3	2	2
0.9990	31.12010	46.68015	0.131568	0.9980	46.48280	94.4	94.3	2	2
0.9950	13.63602	20.45404	0.166939	0.9900	20.20363	94.5	94.3	2	2
0.9900	9.491222	14.23683	0.188829	0.9800	13.95359	94.6	94.3	2	2
0.9750	5.810220	8.715330	0.228534	0.9500	8.372529	94.7	94.3	2	2
0.9500	3.951067	5.926601	0.272025	0.9000	5.518564	95.0	94.3	2	2
0.9000	2.630676	3.946014	0.336730	0.8000	3.440919	95.3	94.3	2	2
0.8000	1.686956	2.530434	0.444770	0.6000	1.863280	95.9	94.3	2	2
0.7000	1.260304	1.890456	0.551900	0.4000	1.062605	96.4	94.3	2	2
0.6000	0.999090	1.498635	0.671202	0.2000	0.491831	97.0	94.4	2	2
0.5000	0.814367	1.221551				97.5		2	2
0.9999	21.04039	38.57404	0.097003	0.9998	38.39620			3	3
0.9995	12.09232	22.16925	0.114951	0.9990	21.95851	90.5	90.3	3	3
0.9990	9.491222	17.40057	0.124906	0.9980	17.17158	90.5	90.2	3	3
0.9950	5.332417	9.776098	0.156366	0.9900	9.489427	90.8	90.2	3	3
0.9900	4.121389	7.555880	0.175425	0.9800	7.234267	91.1	90.2	3	3
0.9750	2.891186	5.300508	0.209245	0.9500	4.916891	91.5	90.2	3	3
0.9500	2.176260	3.989809	0.245258	0.9000	3.540170	92.0	90.2	3	3
0.9000	1.602776	2.938422	0.297045	0.8000	2.393839	92.6	90.2	3	3
0.8000	1.137652	2.085695	0.379433	0.6000	1.390068	93.7	90.2	3	3
0.7000	0.903387	1.656209	0.456955	0.4000	0.818458	94.6	90.1	3	3
0.6000	0.749139	1.373422	0.539296	0.2000	0.384713	95.6	90.2	3	3
0.5000	0.633542	1.161494				96.6		3	3
0.9999	9.491222	19.77338	0.094370	0.9998	19.57677			4	4
0.9995	6.173911	12.86232	0.111272	0.9990	12.63050	87.6	86.9	4	4
0.9990	5.107107	10.63981	0.120574	0.9980	10.38861	87.7	86.9	4	4
0.9950	3.234883	6.739340	0.149639	0.9900	6.427591	88.3	86.9	4	4
0.9900	2.630676	5.480575	0.167009	0.9800	5.132640	88.7	86.9	4	4
0.9750	1.972806	4.110012	0.197406	0.9500	3.698749	89.3	86.9	4	4
0.9500	1.561744	3.253634	0.229206	0.9000	2.776123	90.0	86.9	4	4
0.9000	1.210191	2.521231	0.273987	0.8000	1.950424	90.9	86.8	4	4
0.8000	0.905098	1.885621	0.343171	0.6000	1.170681	92.4	86.8	4	4
0.7000	0.742174	1.546197	0.406260	0.4000	0.699822	93.6	86.8	4	4
0.6000	0.630490	1.313522	0.471432	0.2000	0.331371	94.8	86.8	4	4
0.5000	0.544010	1.133355				96.1		4	4

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	5.795201	13.23238	0.092424	0.9998	13.02134			5	5
0.9995	4.052508	9.253226	0.108576	0.9990	9.005311	85.4	84.2	5	5
0.9990	3.457000	7.893482	0.117415	0.9980	7.625385	85.7	84.2	5	5
0.9950	2.350046	5.365938	0.144807	0.9900	5.035296	86.4	84.2	5	5
0.9900	1.969761	4.497620	0.161015	0.9800	4.129969	86.9	84.2	5	5
0.9750	1.537455	3.510523	0.189100	0.9500	3.078745	87.7	84.2	5	5
0.9500	1.254847	2.865235	0.218114	0.9000	2.367208	88.6	84.1	5	5
0.9000	1.003167	2.290565	0.258376	0.8000	1.700605	89.7	84.1	5	5
0.8000	0.775084	1.769774	0.319325	0.6000	1.040650	91.4	84.0	5	5
0.7000	0.648595	1.480959	0.373717	0.4000	0.627639	92.9	84.0	5	5
0.6000	0.992163	1.273276	0.754215	0.2000	0.305367*	94.7	82.1	4	5
0.5000	0.862837	1.107308				96.7		4	5
0.9999	4.121389	10.09740	0.090892	0.9998	9.874717			6	6
0.9995	3.022011	7.403928	0.106468	0.9990	7.143080	83.8	82.0	6	6
0.9990	2.630676	6.445157	0.114954	0.9980	6.163518	84.1	82.0	6	6
0.9950	1.874013	4.591332	0.141084	0.9900	4.245676	85.0	82.0	6	6
0.9900	1.602776	3.926801	0.156427	0.9800	3.543554	85.6	81.9	6	6
0.9750	1.285111	3.148522	0.182811	0.9500	2.700634	86.5	81.9	6	6
0.9500	1.070836	2.623549	0.209810	0.9000	2.109515	87.5	81.8	6	6
0.9000	0.874523	2.142582	0.246859	0.8000	1.537777	88.8	81.7	6	6
0.8000	1.160510	1.682740	0.508397	0.6000	0.945564	91.3	82.3	5	6
0.7000	0.979760	1.420652	0.586569	0.4000	0.570127	93.5	82.8	5	6
0.6000	0.852532	1.236171	0.665578	0.2000	0.271083	95.3	83.0	5	6
0.5000	0.751821	1.090141				97.1		5	6
0.9999	3.201607	8.301310	0.089636	0.9998	8.068897			7	7
0.9995	2.427707	6.294698	0.104749	0.9990	6.023097	82.5	80.1	7	7
0.9990	2.143967	5.559001	0.112953	0.9980	5.266130	82.8	80.1	7	7
0.9950	1.579245	4.094756	0.138082	0.9900	3.736729	83.9	80.0	7	7
0.9900	1.370422	3.553309	0.152747	0.9800	3.157258	84.5	79.9	7	7
0.9750	1.120405	2.905051	0.177810	0.9500	2.444014*	85.6	79.8	7	7
0.9500	1.532390	2.440878	0.341250	0.9000	1.897315	87.3	81.1	6	7
0.9000	1.261314	2.009093	0.393616	0.8000	1.382118	89.5	81.9	6	7
0.8000	1.008139	1.605822	0.470785	0.6000	0.855928	92.1	82.7	6	7
0.7000	0.863959	1.376163	0.538095	0.4000	0.519055	94.0	83.0	6	7
0.6000	0.760539	1.211430	0.605166	0.2000	0.247487	95.7	83.1	6	7
0.5000	0.677386	1.078979				97.3		6	7
0.9999	2.630676	7.149802	0.088576	0.9998	6.909065			8	8
0.9995	2.045421	5.559163	0.103304	0.9990	5.278396	81.4	78.4	8	8
0.9990	1.825962	4.962704	0.111275	0.9980	4.660275	81.8	78.3	8	8
0.9950	2.175662	3.737477	0.233223	0.9900	3.336833	83.3	79.3	7	8
0.9900	1.894905	3.255175	0.254500	0.9800	2.817981	84.6	79.9	7	8
0.9750	1.558881	2.677936	0.290142	0.9500	2.179514	86.5	80.8	7	8
0.9500	1.327002	2.279601	0.325660	0.9000	1.720163	88.2	81.5	7	8
0.9000	1.110127	1.907040	0.373262	0.8000	1.265829	90.1	82.0	7	8
0.8000	0.902888	1.551033	0.442520	0.6000	0.790847	92.5	82.6	7	8
0.7000	0.782461	1.344155	0.502148	0.4000	0.481537	94.3	82.8	7	8
0.6000	0.694828	1.193615	0.560911	0.2000	0.230050	95.9	82.9	7	8
0.5000	0.878878	1.070348				97.5		6	8

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	2.245571	6.352650	0.087661	0.9998	6.104659			9	9
0.9995	2.740770	5.012782	0.179101	0.9990	4.685212	80.9	77.9	8	9
0.9990	2.452161	4.484924	0.191088	0.9980	4.135430	81.8	78.5	8	9
0.9950	1.864991	3.411009	0.226624	0.9900	2.996522	84.2	79.7	8	9
0.9900	1.642726	3.004494	0.246683	0.9800	2.553320	85.4	80.2	8	9
0.9750	1.372153	2.509625	0.280082	0.9500	1.997364	87.1	80.9	8	9
0.9500	1.182058	2.161946	0.313124	0.9000	1.589251	88.7	81.4	8	9
0.9000	1.001314	1.831371	0.357057	0.8000	1.178325	90.5	81.8	8	9
0.8000	0.825468	1.509754	0.420313	0.6000	0.741014	92.7	82.2	8	9
0.7000	0.721641	1.319859	0.474202	0.4000	0.452558	94.4	82.4	8	9
0.6000	0.887721	1.179753	0.721851	0.2000	0.220436*	96.0	81.0	7	9
0.5000	0.799401	1.062379				97.7		7	9
0.9999	2.947923	5.686449	0.153739	0.9998	5.389891			9	10
0.9995	2.347536	4.528322	0.175598	0.9990	4.189598	81.8	78.4	9	10
0.9990	2.118425	4.086375	0.187109	0.9980	3.725447	82.7	78.8	9	10
0.9950	1.643984	3.171193	0.221066	0.9900	2.744764	84.8	79.8	9	10
0.9900	1.460964	2.818153	0.240127	0.9800	2.354955	85.9	80.2	9	10
0.9750	1.235110	2.382487	0.271703	0.9500	1.858380	87.5	80.7	9	10
0.9500	1.074125	2.071954	0.302752	0.9000	1.487955	89.0	81.1	9	10
0.9000	0.919013	1.772746	0.343757	0.8000	1.109649	90.7	81.4	9	10
0.8000	0.765884	1.477367	0.402287	0.6000	0.701368	92.9	81.7	9	10
0.7000	0.909329	1.299402	0.607133	0.4000	0.431828*	94.6	81.4	8	10
0.6000	0.816001	1.166040	0.671425	0.2000	0.206594*	96.3	81.5	8	10
0.5000	0.739333	1.056483				97.9		8	10
0.9999	2.550353	5.151400	0.151406	0.9998	4.845579			10	11
0.9995	2.063467	4.167950	0.172563	0.9990	3.819394	82.4	78.5	10	11
0.9990	1.874971	3.787211	0.183669	0.9980	3.416223	83.2	78.9	10	11
0.9950	1.478934	2.987265	0.216293	0.9900	2.550380	85.2	79.6	10	11
0.9900	1.323778	2.673868	0.234518	0.9800	2.200170	86.2	80.0	10	11
0.9750	1.130147	2.282758	0.264576	0.9500	1.748346	87.8	80.4	10	11
0.9500	0.990477	2.000643	0.293977	0.9000	1.406844	89.1	80.6	10	11
0.9000	0.854410	1.725804	0.332584	0.8000	1.054024	90.8	80.9	10	11
0.8000	0.952715	1.448010	0.514005	0.6000	0.666785	93.1	81.4	9	11
0.7000	0.842195	1.280032	0.573667	0.4000	0.408130	95.0	81.7	9	11
0.6000	0.760095	1.155252	0.631482	0.2000	0.195476	96.5	81.8	9	11
0.5000	0.692122	1.051940				98.0		9	11
0.9999	2.255961	4.744761	0.149346	0.9998	4.430656			11	12
0.9995	1.849261	3.889386	0.169894	0.9990	3.532063	82.8	78.5	11	12
0.9990	1.689910	3.554236	0.180650	0.9980	3.174291	83.5	78.7	11	12
0.9950	1.351016	2.841471	0.212129	0.9900	2.395319	85.4	79.4	11	12
0.9900	1.216518	2.558593	0.229641	0.9800	2.075608	86.4	79.6	11	12
0.9750	1.047075	2.202219	0.258411	0.9500	1.658726*	87.9	79.9	11	12
0.9500	0.923619	1.942565	0.286423	0.9000	1.340157*	89.2	80.1	11	12
0.9000	1.049144	1.681999	0.426071	0.8000	0.998918	91.1	81.1	10	12
0.8000	0.886772	1.421683	0.491465	0.6000	0.633760	93.5	81.5	10	12
0.7000	0.788618	1.264321	0.546209	0.4000	0.388632	95.2	81.7	10	12
0.6000	0.715134	1.146511	0.598927	0.2000	0.186305	96.7	81.9	10	12
0.5000	0.825023	1.047678				98.2		9	12

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	2.029881	4.425411	0.147507	0.9998	4.103827			12	13
0.9995	1.682235	3.667497	0.167520	0.9990	3.302281	83.0	78.3	12	13
0.9990	1.544630	3.367500	0.177970	0.9980	2.979503	83.7	78.5	12	13
0.9950	1.248943	2.722862	0.208452	0.9900	2.268410*	85.6	79.0	12	13
0.9900	1.130291	2.464186	0.225345	0.9800	1.972903*	86.5	79.2	12	13
0.9750	1.268036	2.130470	0.334656	0.9500	1.568203	88.2	80.2	11	13
0.9500	1.121884	1.884915	0.368006	0.9000	1.266617	89.7	80.7	11	13
0.9000	0.978212	1.643527	0.411445	0.8000	0.952245	91.5	81.1	11	13
0.8000	0.833211	1.399906	0.472453	0.6000	0.606121	93.7	81.5	11	13
0.7000	0.744749	1.251278	0.523193	0.4000	0.372244	95.4	81.7	11	13
0.6000	0.678079	1.139263	0.571790	0.2000	0.178579	96.8	81.8	11	13
0.5000	0.775205	1.044046				98.3		10	13
0.9999	1.851142	4.167962	0.145849	0.9998	3.839573			13	14
0.9995	1.548457	3.486447	0.165387	0.9990	3.114068	83.2	78.0	13	14
0.9990	1.427597	3.214323	0.175566	0.9980	2.819026*	83.8	78.1	13	14
0.9950	1.495147	2.618843	0.273156	0.9900	2.140393	85.8	79.3	12	14
0.9900	1.355763	2.374704	0.293296	0.9800	1.860978	86.9	79.8	12	14
0.9750	1.178786	2.064717	0.326094	0.9500	1.493543	88.6	80.3	12	14
0.9500	1.048767	1.836981	0.357757	0.9000	1.210348	90.0	80.7	12	14
0.9000	0.919932	1.611318	0.398814	0.8000	0.912770	91.8	81.1	12	14
0.8000	0.788749	1.381542	0.456148	0.6000	0.582570	93.9	81.4	12	14
0.7000	0.708079	1.240245	0.503560	0.4000	0.358228	95.5	81.5	12	14
0.6000	0.798285	1.132151	0.676158	0.2000	0.173204*	97.0	81.0	11	14
0.5000	0.734080	1.041094				98.4		11	14
0.9999	1.706449	3.955939	0.144344	0.9998	3.621317*			14	15
0.9995	1.833879	3.334413	0.220180	0.9990	2.934075	83.3	78.2	13	15
0.9990	1.692871	3.078027	0.232450	0.9980	2.655380	84.1	78.6	13	15
0.9950	1.387217	2.522278	0.267782	0.9900	2.035389	86.3	79.4	13	15
0.9900	1.263430	2.297205	0.287130	0.9800	1.775137	87.3	79.8	13	15
0.9750	1.105156	2.009427	0.318540	0.9500	1.430248	88.9	80.3	13	15
0.9500	0.988009	1.796427	0.348749	0.9000	1.162322	90.3	80.6	13	15
0.9000	0.871119	1.583895	0.387765	0.8000	0.878849	91.9	80.9	13	15
0.8000	0.751176	1.365811	0.441972	0.6000	0.562205	94.0	81.2	13	15
0.7000	0.828123	1.229676	0.594566	0.4000	0.346808*	95.6	81.1	12	15
0.6000	0.758287	1.125977	0.646176	0.2000	0.166473*	97.1	81.2	12	15
0.5000	0.841029	1.038583				98.5		11	15
0.9999	2.005152	3.771148	0.193769	0.9998	3.406720			14	16
0.9995	1.695043	3.187916	0.216892	0.9990	2.780002	83.8	78.4	14	16
0.9990	1.570296	2.953301	0.228792	0.9980	2.523005	84.6	78.7	14	16
0.9950	1.297775	2.440763	0.262963	0.9900	1.946201	86.6	79.5	14	16
0.9900	1.186488	2.231462	0.281616	0.9800	1.701819	87.6	79.8	14	16
0.9750	1.043321	1.962204	0.311808	0.9500	1.375778	89.2	80.2	14	16
0.9500	0.936661	1.761606	0.340749	0.9000	1.120750	90.5	80.4	14	16
0.9000	0.829582	1.560218	0.377993	0.8000	0.849315	92.1	80.7	14	16
0.8000	0.872865	1.350668	0.521712	0.6000	0.543372	94.2	81.1	13	16
0.7000	0.788504	1.220128	0.572361	0.4000	0.334458	95.8	81.3	13	16
0.6000	0.724240	1.120686	0.620435	0.2000	0.160627	97.2	81.4	13	16
0.5000	0.798618	1.036124				98.6		12	16

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	1.855646	3.599122	0.191376	0.9998	3.227938			15	17
0.9995	1.579822	3.064148	0.213903	0.9990	2.649271	84.2	78.5	15	17
0.9990	1.468123	2.847503	0.225474	0.9980	2.410184	84.9	78.7	15	17
0.9950	1.222420	2.370949	0.258609	0.9900	1.869363	86.9	79.4	15	17
0.9900	1.121347	2.174911	0.276643	0.9800	1.638347	87.9	79.6	15	17
0.9750	0.990611	1.921342	0.305758	0.9500	1.328309*	89.3	80.0	15	17
0.9500	0.892645	1.731332	0.333581	0.9000	1.084334*	90.6	80.2	15	17
0.9000	0.957399	1.537792	0.447099	0.8000	0.819653	92.3	80.8	14	17
0.8000	0.832352	1.336939	0.505315	0.6000	0.525293	94.4	81.2	14	17
0.7000	0.754460	1.211827	0.552980	0.4000	0.323621	95.9	81.3	14	17
0.6000	0.694857	1.116092	0.598052	0.2000	0.155490	97.3	81.4	14	17
0.5000	0.762453	1.034053				98.6		13	17
0.9999	1.730698	3.452929	0.189181	0.9998	3.075491			16	18
0.9995	1.482686	2.958119	0.211170	0.9990	2.536812	84.5	78.4	16	18
0.9990	1.381653	2.756547	0.222443	0.9980	2.312749	85.2	78.7	16	18
0.9950	1.158040	2.310414	0.254647	0.9900	1.802366	87.1	79.2	16	18
0.9900	1.065450	2.125688	0.272127	0.9800	1.582765*	88.1	79.5	16	18
0.9750	1.134481	1.885251	0.363976	0.9500	1.280405	89.5	80.1	15	18
0.9500	1.024211	1.702008	0.395328	0.9000	1.045061	90.9	80.5	15	18
0.9000	0.912925	1.517076	0.435503	0.8000	0.793368	92.5	80.8	15	18
0.8000	0.797279	1.324897	0.490796	0.6000	0.509305	94.6	81.1	15	18
0.7000	0.724847	1.204533	0.535884	0.4000	0.314015	96.1	81.3	15	18
0.6000	0.787487	1.111754	0.679805	0.2000	0.152023*	97.4	80.8	14	18
0.5000	0.731197	1.032285				98.7		14	18
0.9999	1.624764	3.327094	0.187158	0.9998	2.943843			17	19
0.9995	1.399692	2.866204	0.208657	0.9990	2.438929	84.7	78.4	17	19
0.9990	1.307514	2.677448	0.219659	0.9980	2.227644	85.4	78.6	17	19
0.9950	1.102371	2.257369	0.251020	0.9900	1.743345*	87.3	79.0	17	19
0.9900	1.213796	2.080939	0.324908	0.9800	1.523915	88.3	79.7	16	19
0.9750	1.078882	1.849642	0.356544	0.9500	1.238382	89.8	80.2	16	19
0.9500	0.977280	1.675455	0.386644	0.9000	1.012590	91.1	80.5	16	19
0.9000	0.874242	1.498807	0.425100	0.8000	0.770012	92.7	80.8	16	19
0.8000	0.766582	1.314233	0.477829	0.6000	0.495039	94.7	81.1	16	19
0.7000	0.818087	1.198012	0.608666	0.4000	0.306678*	96.1	80.9	15	19
0.6000	0.756412	1.107695	0.655755	0.2000	0.147403*	97.5	80.9	15	19
0.5000	0.815108	1.030628				98.8		14	19
0.9999	1.533835	3.217587	0.185284	0.9998	2.828909			18	20
0.9995	1.327954	2.785702	0.206334	0.9990	2.352867*	84.9	78.2	18	20
0.9990	1.243230	2.607974	0.217089	0.9980	2.152578*	85.6	78.4	18	20
0.9950	1.250475	2.206345	0.300207	0.9900	1.676658	87.6	79.5	17	20
0.9900	1.155046	2.037970	0.319249	0.9800	1.474685	88.6	79.8	17	20
0.9750	1.030490	1.818203	0.349790	0.9500	1.201032	90.0	80.2	17	20
0.9500	0.936249	1.651924	0.378773	0.9000	0.983615	91.3	80.5	17	20
0.9000	0.840256	1.482553	0.415701	0.8000	0.749088	92.9	80.7	17	20
0.8000	0.739461	1.304709	0.466161	0.6000	0.482212	94.8	81.0	17	20
0.7000	0.786774	1.191496	0.590012	0.4000	0.297978*	96.3	81.0	16	20
0.6000	0.729065	1.104100	0.634461	0.2000	0.143269*	97.6	81.1	16	20
0.5000	0.782936	1.029095				98.8		15	20

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	1.633258	3.033743	0.222895	0.9998	2.619720			19	22
0.9995	1.422825	2.642869	0.246148	0.9990	2.185654	85.5	78.6	19	22
0.9990	1.335863	2.481339	0.257958	0.9980	2.002187	86.2	78.9	19	22
0.9950	1.140511	2.118477	0.291350	0.9900	1.577301	88.1	79.5	19	22
0.9900	1.058338	1.965841	0.309288	0.9800	1.391344	89.0	79.7	19	22
0.9750	0.950269	1.765106	0.337944	0.9500	1.137381*	90.4	80.0	19	22
0.9500	0.867838	1.611991	0.365016	0.9000	0.933982*	91.6	80.3	19	22
0.9000	0.904556	1.454055	0.462181	0.8000	0.711108	93.1	80.7	18	22
0.8000	0.800424	1.286666	0.515348	0.6000	0.458254	95.1	81.0	18	22
0.7000	0.734334	1.180427	0.558256	0.4000	0.283041	96.5	81.1	18	22
0.6000	0.780078	1.097945	0.682688	0.2000	0.137075*	97.7	80.6	17	22
0.5000	0.729377	1.026583				98.9		17	22
0.9999	1.480046	2.875175	0.218376	0.9998	2.450953			21	24
0.9995	1.299520	2.524480	0.240663	0.9990	2.056962	86.0	78.6	21	24
0.9990	1.224348	2.378449	0.251950	0.9980	1.889004	86.6	78.8	21	24
0.9950	1.054133	2.047785	0.283750	0.9900	1.496564*	88.4	79.3	21	24
0.9900	0.981925	1.907512	0.300767	0.9800	1.323235*	89.3	79.5	21	24
0.9750	1.016340	1.720282	0.378702	0.9500	1.079282	90.7	80.2	20	24
0.9500	0.931525	1.576723	0.407375	0.9000	0.887190	92.0	80.4	20	24
0.9000	0.844192	1.428900	0.443635	0.8000	0.677992	93.4	80.7	20	24
0.8000	0.751366	1.271780	0.492735	0.6000	0.437765	95.3	80.9	20	24
0.7000	0.784638	1.171171	0.602885	0.4000	0.271289*	96.6	80.8	19	24
0.6000	0.731876	1.092416	0.644422	0.2000	0.130537*	97.8	80.9	19	24
0.5000	0.772545	1.024363				99.0		18	24
0.9999	1.359356	2.747376	0.214399	0.9998	2.314058*			23	26
0.9995	1.201441	2.428217	0.235854	0.9990	1.951535*	86.3	78.5	23	26
0.9990	1.135260	2.294459	0.246693	0.9980	1.795872*	86.9	78.7	23	26
0.9950	1.121519	1.986307	0.320166	0.9900	1.419264	88.8	79.6	22	26
0.9900	1.047435	1.855098	0.338253	0.9800	1.256022	89.7	79.8	22	26
0.9750	0.949123	1.680979	0.366976	0.9500	1.031032	91.1	80.1	22	26
0.9500	0.873405	1.546876	0.393935	0.9000	0.849183	92.3	80.4	22	26
0.9000	0.794974	1.407967	0.427895	0.8000	0.650128*	93.7	80.6	22	26
0.8000	0.801204	1.258761	0.533849	0.6000	0.420039	95.4	80.8	21	26
0.7000	0.740143	1.162829	0.574802	0.4000	0.259766	96.8	80.9	21	26
0.6000	0.692415	1.087843	0.612826	0.2000	0.125041	97.9	81.0	21	26
0.5000	0.810547	1.022556				99.1		19	26
0.9999	1.431924	2.640235	0.245342	0.9998	2.187865			24	28
0.9995	1.270901	2.343335	0.268334	0.9990	1.848572	86.7	78.9	24	28
0.9990	1.203231	2.218563	0.279909	0.9980	1.702456	87.4	79.1	24	28
0.9950	1.048515	1.933291	0.312305	0.9900	1.357451	89.2	79.6	24	28
0.9900	0.982206	1.811029	0.329524	0.9800	1.203439*	90.0	79.7	24	28
0.9750	0.893769	1.647964	0.356792	0.9500	0.990097*	91.3	80.0	24	28
0.9500	0.925437	1.521269	0.429916	0.9000	0.814556	92.5	80.4	23	28
0.9000	0.845026	1.389086	0.465383	0.8000	0.624073	93.9	80.6	23	28
0.8000	0.758737	1.247240	0.513062	0.6000	0.403849	95.6	80.9	23	28
0.7000	0.703098	1.155778	0.551063	0.4000	0.249921	96.9	81.0	23	28
0.6000	0.733611	1.083670	0.651911	0.2000	0.120686*	98.0	80.8	22	28
0.5000	0.765155	1.020957				99.1		21	28

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	1.331244	2.544877	0.240983	0.9998	2.084201			26	30
0.9995	1.187769	2.270603	0.263146	0.9990	1.767559*	87.1	78.8	26	30
0.9990	1.127161	2.154742	0.274279	0.9980	1.630415*	87.7	79.0	26	30
0.9950	0.987834	1.888396	0.305356	0.9900	1.304662*	89.4	79.4	26	30
0.9900	1.035766	1.772873	0.361724	0.9800	1.153728	90.3	79.9	25	30
0.9750	0.945340	1.618095	0.390291	0.9500	0.950052	91.6	80.2	25	30
0.9500	0.875155	1.497963	0.416965	0.9000	0.784262	92.7	80.4	25	30
0.9000	0.801920	1.372610	0.450391	0.8000	0.601696	94.1	80.6	25	30
0.8000	0.800851	1.237304	0.548396	0.6000	0.390039*	95.8	80.7	24	30
0.7000	0.743887	1.149296	0.587613	0.4000	0.241441*	97.0	80.8	24	30
0.6000	0.699093	1.080089	0.623834	0.2000	0.116274*	98.1	80.9	24	30
0.5000	0.798347	1.019592				99.2		22	30
0.9999	1.247588	2.464188	0.237062	0.9998	1.995952*			28	32
0.9995	1.118229	2.208683	0.258492	0.9990	1.698120*	87.4	78.7	28	32
0.9990	1.182691	2.099468	0.303082	0.9980	1.561448	88.0	79.2	27	32
0.9950	1.040594	1.847223	0.335733	0.9900	1.251242	89.7	79.7	27	32
0.9900	0.979187	1.738215	0.352999	0.9800	1.111586	90.6	79.9	27	32
0.9750	0.896771	1.591914	0.380224	0.9500	0.916956*	91.8	80.1	27	32
0.9500	0.832521	1.477859	0.405574	0.9000	0.757899*	92.9	80.3	27	32
0.9000	0.844329	1.358098	0.482919	0.8000	0.581326	94.3	80.6	26	32
0.8000	0.763489	1.228068	0.529213	0.6000	0.376831	95.9	80.8	26	32
0.7000	0.710993	1.143629	0.565898	0.4000	0.233385	97.1	80.9	26	32
0.6000	0.734721	1.076834	0.657776	0.2000	0.112773*	98.2	80.7	25	32
0.5000	0.759630	1.018388				99.2		24	32
0.9999	1.304359	2.393338	0.263372	0.9998	1.910083			29	34
0.9995	1.172722	2.151800	0.285959	0.9990	1.627101	87.7	79.0	29	34
0.9990	1.116753	2.049105	0.297258	0.9980	1.503673	88.4	79.2	29	34
0.9950	0.987201	1.811391	0.328650	0.9900	1.208360*	90.0	79.6	29	34
0.9900	0.930945	1.708170	0.345207	0.9800	1.074758*	90.8	79.8	29	34
0.9750	0.940470	1.568902	0.409567	0.9500	0.885658	92.0	80.2	28	34
0.9500	0.874911	1.459535	0.435872	0.9000	0.732408	93.1	80.4	28	34
0.9000	0.806087	1.344723	0.468692	0.8000	0.562847	94.4	80.6	28	34
0.8000	0.731322	1.219998	0.512404	0.6000	0.365201	96.0	80.8	28	34
0.7000	0.746381	1.138494	0.597874	0.4000	0.226525*	97.2	80.8	27	34
0.6000	0.704060	1.073941	0.632517	0.2000	0.109129*	98.3	80.8	27	34
0.5000	0.789085	1.017321				99.3		25	34
0.9999	1.231646	2.329321	0.259203	0.9998	1.839110*			31	36
0.9995	1.111570	2.102229	0.281069	0.9990	1.570665*	88.0	79.0	31	36
0.9990	1.060322	2.005308	0.291989	0.9980	1.453091*	88.6	79.1	31	36
0.9950	1.031931	1.779626	0.355450	0.9900	1.166631	90.2	79.8	30	36
0.9900	0.974643	1.680829	0.372669	0.9800	1.038140	91.0	79.9	30	36
0.9750	0.897353	1.547539	0.399724	0.9500	0.858190	92.3	80.2	30	36
0.9500	0.836759	1.443040	0.424820	0.9000	0.710412	93.3	80.4	30	36
0.9000	0.772922	1.332950	0.456054	0.8000	0.546458*	94.6	80.5	30	36
0.8000	0.766625	1.212573	0.542415	0.6000	0.354634	96.1	80.7	29	36
0.7000	0.716834	1.133818	0.577885	0.4000	0.219776	97.3	80.8	29	36
0.6000	0.735442	1.071320	0.662513	0.2000	0.106235*	98.3	80.6	28	36
0.5000	0.755342	1.016381				99.3		27	36

TABLE 1 (CONTINUED)

LOWEST UPPER CONFIDENCE BOUNDS, BASED ON ONE ORDER STATISTIC, FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9999	1.169119	2.273432	0.255393	0.9998	1.776802*			33	38
0.9995	1.157737	2.058343	0.305457	0.9990	1.515270	88.2	79.2	32	38
0.9990	1.105670	1.965774	0.316842	0.9980	1.402460	88.9	79.3	32	38
0.9950	0.984449	1.750255	0.348350	0.9900	1.130922	90.5	79.7	32	38
0.9900	0.931487	1.656092	0.364904	0.9800	1.007329*	91.3	79.9	32	38
0.9750	0.859813	1.528664	0.390866	0.9500	0.833742*	92.4	80.1	32	38
0.9500	0.873587	1.428354	0.451754	0.9000	0.689716	93.5	80.4	31	38
0.9000	0.808551	1.322017	0.483943	0.8000	0.530748	94.7	80.6	31	38
0.8000	0.737490	1.205828	0.526622	0.6000	0.344778	96.3	80.7	31	38
0.7000	0.748075	1.129627	0.606308	0.4000	0.214075*	97.4	80.7	30	38
0.6000	0.707875	1.068923	0.639558	0.2000	0.103161*	98.4	80.7	30	38
0.5000	0.781813	1.015523				99.3		28	38
0.9999	1.215558	2.222699	0.278301	0.9998	1.713814			34	40
0.9995	1.103409	2.017630	0.300435	0.9990	1.468273	88.5	79.2	34	40
0.9990	1.055310	1.929679	0.311453	0.9980	1.360173*	89.1	79.3	34	40
0.9950	0.942943	1.724213	0.341893	0.9900	1.099047*	90.7	79.7	34	40
0.9900	0.893670	1.634113	0.357852	0.9800	0.979765*	91.4	79.8	34	40
0.9750	0.896554	1.511309	0.416292	0.9500	0.809570	92.6	80.2	33	40
0.9500	0.839140	1.414526	0.441081	0.9000	0.671003	93.6	80.4	33	40
0.9000	0.778377	1.312100	0.471827	0.8000	0.516747	94.9	80.6	33	40
0.8000	0.768717	1.199726	0.553447	0.6000	0.335968*	96.4	80.7	32	40
0.7000	0.721278	1.125689	0.587802	0.4000	0.208314*	97.5	80.8	32	40
0.6000	0.735911	1.066759	0.666431	0.2000	0.100717*	98.4	80.5	31	40
0.5000	0.806226	1.014769				99.4		29	40

*This value of the expected length of the confidence interval (ELCI) is not as small as that for a different value of m. See Table 2 for results for that value of m which minimizes the ELCI.

Explanation of Column Headings in Tables 1 and 2

1-P = Level of confidence associated with upper and lower confidence bounds

UCB/x_m = Coefficient of $m^{\underline{th}}$ order statistic in the upper confidence bound

EVUCB = Expected value of the upper confidence bound

LCB/x_m = Coefficient of $m^{\underline{th}}$ order statistic in the lower confidence bound

1-2P = Level of confidence associated with the central confidence interval

ELCI = Expected length of the central confidence interval

EFUB (%) = Effectiveness (in percent) of the upper confidence bound

EFI(%) = Effectiveness (in percent) of the central confidence interval

m = Order of the order statistic on which confidence bounds are based

n = Size of sample

TABLE 2

SHORTEST CENTRAL CONFIDENCE INTERVALS,
BASED ON DIFFERENT ORDER STATISTIC THAN LOWEST UPPER CONFIDENCE BOUNDS,
FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.6000	0.559569	1.277683	0.428851	0.2000	0.298473	94.4	84.0	5	5
0.9750	1.828384	2.912355	0.302555	0.9500	2.430428	85.4	80.3	6	7
0.6000	0.645221	1.180089	0.526836	0.2000	0.216523	96.0	82.5	8	9
0.7000	0.674289	1.300682	0.451711	0.4000	0.429347	94.5	81.8	9	10
0.6000	0.606238	1.169415	0.499627	0.2000	0.205650	96.0	81.9	9	10
0.9750	1.378577	2.210149	0.344476	0.9500	1.657881	87.6	80.0	10	12
0.9500	1.211687	1.942589	0.379810	0.9000	1.333673	89.2	80.5	10	12
0.9950	1.627988	2.735238	0.279206	0.9900	2.266134	85.2	79.0	11	13
0.9900	1.468685	2.467587	0.300255	0.9800	1.963118	86.4	79.6	11	13
0.9990	1.842593	3.227417	0.236511	0.9980	2.813154	83.5	78.3	12	14
0.6000	0.646930	1.133138	0.548755	0.2000	0.171959	96.9	81.6	12	14
0.9999	2.187085	3.976622	0.196396	0.9998	3.619530			13	15
0.7000	0.676905	1.230769	0.486571	0.4000	0.346070	95.6	81.3	13	15
0.6000	0.620320	1.127883	0.528907	0.2000	0.166208	96.9	81.4	13	15
0.9750	1.199072	1.925972	0.372209	0.9500	1.328123	89.1	80.0	14	17
0.9500	1.078461	1.732245	0.404975	0.9000	1.081767	90.6	80.4	14	17
0.9900	1.281664	2.129836	0.331117	0.9800	1.579595	87.9	79.6	15	18
0.6000	0.669201	1.112061	0.578375	0.2000	0.150932	97.4	81.3	15	18
0.9950	1.317612	2.258922	0.305226	0.9900	1.735639	87.2	79.4	16	19
0.7000	0.698820	1.198062	0.520667	0.4000	0.305427	96.1	81.2	16	19
0.6000	0.646573	1.108489	0.560915	0.2000	0.146852	97.4	81.3	16	19
0.9995	1.582632	2.792405	0.252492	0.9990	2.346906	84.7	78.4	17	20
0.9990	1.479642	2.610690	0.264921	0.9980	2.143261	85.5	78.7	17	20
0.7000	0.675737	1.192274	0.507016	0.4000	0.297692	96.2	81.1	17	20
0.6000	0.626441	1.105297	0.545295	0.2000	0.143175	97.5	81.1	17	20
0.9750	1.099866	1.768013	0.392402	0.9500	1.137234	90.3	80.0	18	22
0.9500	1.003299	1.612783	0.423138	0.9000	0.932598	91.6	80.4	18	22
0.6000	0.683061	1.098006	0.598356	0.2000	0.136161	97.7	81.2	18	22
0.9950	1.211151	2.050025	0.329163	0.9900	1.492876	88.3	79.5	20	24
0.9900	1.127156	1.907852	0.348266	0.9800	1.318368	89.3	79.8	20	24
0.7000	0.692030	1.171347	0.532142	0.4000	0.270630	96.6	81.0	20	24
0.6000	0.645753	1.093018	0.568803	0.2000	0.130248	97.8	81.1	20	24
0.9999	1.555456	2.754846	0.250229	0.9998	2.311669			22	26
0.9995	1.372122	2.430146	0.274169	0.9990	1.944570	86.2	78.8	22	26
0.9990	1.295533	2.294500	0.286250	0.9980	1.787526	86.9	79.0	22	26
0.9000	0.896438	1.408382	0.482693	0.8000	0.650029	93.6	80.6	21	26

TABLE 2 (CONTINUED)

SHORTEST CENTRAL CONFIDENCE INTERVALS,
BASED ON DIFFERENT ORDER STATISTIC THAN LOWEST UPPER CONFIDENCE BOUNDS,
FOR PARAMETER OF 1-PARAMETER NEGATIVE EXPONENTIAL POPULATION

1-P	UCB/x_m	EVUCB	LCB/x_m	1-2P	ELCI	EFUB(%)	EFI(%)	m	n
0.9900	1.103136	1.813376	0.371585	0.9800	1.202550	89.9	79.8	23	28
0.9750	1.002869	1.648554	0.401705	0.9500	0.988216	91.3	80.2	23	28
0.6000	0.659424	1.083987	0.586217	0.2000	0.120341	98.0	81.0	23	28
0.9995	1.329477	2.275604	0.297361	0.9990	1.766625	86.9	78.9	25	30
0.9990	1.260820	2.158087	0.309568	0.9980	1.628213	87.6	79.1	25	30
0.9950	1.103420	1.888673	0.343651	0.9900	1.300461	89.4	79.7	25	30
0.8000	0.722905	1.237362	0.495148	0.6000	0.389841	95.8	80.8	25	30
0.7000	0.671704	1.149724	0.530678	0.4000	0.241387	97.0	80.9	25	30
0.6000	0.631367	1.080681	0.563442	0.2000	0.116264	98.1	80.9	25	30
0.9999	1.390009	2.467492	0.267963	0.9998	1.991814			27	32
0.9995	1.244357	2.208935	0.291357	0.9990	1.691730	87.4	79.0	27	32
0.9750	0.990566	1.593320	0.420589	0.9500	0.916804	91.8	80.1	26	32
0.9500	0.919078	1.478333	0.448278	0.9000	0.757280	92.9	80.4	26	32
0.6000	0.669571	1.077001	0.599678	0.2000	0.112422	98.2	80.9	26	32
0.9950	1.086829	1.813058	0.363308	0.9900	1.206985	89.9	79.7	28	34
0.9900	1.024403	1.708919	0.381275	0.9800	1.072872	90.8	79.9	28	34
0.7000	0.682572	1.138673	0.546928	0.4000	0.226282	97.2	80.8	28	34
0.6000	0.643987	1.074305	0.578633	0.2000	0.109024	98.2	80.9	28	34
0.9999	1.353606	2.334374	0.287349	0.9998	1.838823			30	36
0.9995	1.220349	2.104563	0.310958	0.9990	1.568298	87.9	79.1	30	36
0.9990	1.163585	2.006672	0.322749	0.9980	1.450071	88.5	79.3	30	36
0.9000	0.842850	1.333137	0.497436	0.8000	0.546342	94.6	80.6	29	36
0.6000	0.677373	1.071402	0.610420	0.2000	0.105899	98.3	80.9	29	36
0.9999	1.279532	2.274881	0.282624	0.9998	1.772403			32	38
0.9900	1.013717	1.657472	0.397906	0.9800	1.006878	91.2	79.9	31	38
0.9750	0.935225	1.529135	0.425858	0.9500	0.832838	92.4	80.2	31	38
0.7000	0.690910	1.129669	0.560183	0.4000	0.213744	97.4	80.8	31	38
0.6000	0.653898	1.069152	0.590896	0.2000	0.103010	98.4	80.8	31	38
0.9990	1.146099	1.931963	0.339744	0.9980	1.359262	89.0	79.4	33	40
0.9950	1.023220	1.724827	0.372365	0.9900	1.097137	90.6	79.8	33	40
0.9900	0.969437	1.634167	0.389479	0.9800	0.977627	91.4	80.0	33	40
0.8000	0.711757	1.199799	0.512485	0.6000	0.335910	96.3	80.7	33	40
0.7000	0.667951	1.125956	0.544375	0.4000	0.208312	97.4	80.8	33	40
0.6000	0.683540	1.066791	0.619208	0.2000	0.100401	98.4	80.8	32	40

See last page of Table 1 for explanation of column headings.

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FIDUCIAL INFERENCE

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A large and important branch of statistical theory is concerned with the testing of hypotheses and the estimation of parameters, but in spite of all the work that has been done in the last thirty years it remains true that statisticians do not entirely agree with each other on the logical basis of statistical inference. One has only to glance at Sir Ronald Fisher's recent (1956) book on Statistical Methods and Scientific Inference (Reference [1]) for evidence of this - he made no attempt to disguise his low opinion of the theory of confidence intervals as developed by J. Neyman and E. S. Pearson. He himself developed the rival theory of fiducial inference, starting back in 1930, but his ideas never became really popular. His own standpoint has not always been quite consistent and his presentation is not always as lucid as it might be, so that most practising statisticians have felt happier with confidence intervals than with fiducial intervals. Quite recently, however, there has been a resurgence of interest in fiducial inference, and an attempt to place it on a firmer mathematical basis. This is the excuse for the present brief discussion of the problem of inference or inductive reasoning.

The first attempt at reasoning inversely and mathematically from effects to causes was made by an English clergyman, Thomas Bayes, and published posthumously in 1763 (Reference [2]). Suppose that in a series of independent trials of an event which is always either a success or a failure, the (constant) probability of a success is θ. Then, if θ is known, the probability of exactly x successes in N trials is given by the well-known binomial law as $\binom{N}{x} \theta^x (1 - \theta)^{N-x}$. If θ is not known, an unbiased estimate of it is x/N, but Bayes realized that we might have prior information (that is, prior to conducting the series of trials) about the value of θ. This information will, at least in theory, enable us to assign to θ a prior probability density $f(\theta)$, but if so the posterior probability density (*after* the results of the trials are known) will depend on both x and $f(\theta)$. In fact, it

is given by:

$$f(\theta|x) = \frac{\theta^{x} (1-\theta)^{N-x} f(\theta)}{\int_0^1 \theta^{x} (1-\theta)^{N-x} f(\theta)\, d\theta} \tag{1}$$

This is a special case of the rule now generally known as Bayes' rule. Bayes suggested, although apparently with some reservations, that in the absence of any other prior information about θ one might reasonably assume that all possible values were equally likely. This amounts to putting $f(\theta) = 1$ in equation (1). The posterior probability density is then

$$f(\theta|x) = \frac{(N+1)!}{x!\ (N-x)!} \theta^{x} (1-\theta)^{N-x} \tag{2}$$

and this is a maximum over θ for $\theta = x/N$.

Bayes' rule itself is unexceptionable. It is the postulate of constant prior probability that has been the target of so much criticism, particularly in recent times. As Fisher has pointed out, we have no more logical reason to assume that θ is constant over the interval 0 to 1 than to assume the same thing for θ^2 or arc sin $(2\theta-1)$ or some other convenient function of θ. Each choice will give rise to its own posterior probability density for θ, so that unless there are good physical reasons for assuming that one variable rather than another has a uniform distribution in a particular problem, Bayes' postulate is an unsafe guide for inference.

Fiducial inference, as distinguished from Bayes inference, makes no reference to prior probability. It does, however, consider θ as a quantity about which meaningful probability statements can be made, probability being regarded as

a measure of rational belief. Before the observations have been made we can say nothing about θ. If X is a statistic (a function of the observations, in our example the number of successes in N trials) and we find that $X = x$ we can then make a fiducial statement about θ. For instance, if $N = 100$ and $x = 40$ we can say that, with probability 0.95, θ lies between the limits 0.303 and 0.503. (These figures are obtained by interpolation in a table of cumulative binomial probability (Reference [3]) or directly from Mainland's table (Reference [4]). It should be noted, however, that Fisher preferred to restrict fiducial inference to situations in which the variate X is continuous rather than discrete.)

In general, if T is a statistic with a known distribution function F depending on θ,

$$P(T \leq t_k) = F(t_k \mid \theta) = k \tag{3}$$

where k is a fixed fraction, say 0.95. On the frequency interpretation of probability this means that in the fraction k of a large number of random samples from a population with parameter θ the value of the measured statistic T will not exceed t_k. Obviously t_k is a function of θ for given k, say $K(\theta)$, and if $K(\theta)$ is monotonic, θ will be a function of t_k, say $K^{-1}(t_k)$. We can then write either

$$P\Big(T \leq K(\theta)\Big) = k \tag{4}$$

or

$$P\Big(\theta \geq K^{-1}(t_k)\Big) = k \tag{5}$$

assuming that $K(\theta)$ is (as in the binomial example) an increasing function of θ. The fiducial viewpoint is expressed by equation (5) in which the probability statement is made about θ, t_k being regarded as known from observation.

From the point of view of the usual theory of statistical inference, developed mainly by Neyman and Pearson, equation (4) is preferable to (5). Nature, according to these authors, presents us with a particular, although unknown, value of θ in the population we are examining. This parameter is not a random variable and probability statements should not therefore be made about it. However, the statistic T which we actually measure (or calculate from the observations) *is* a random variable, varying from sample to sample. It is quite proper, therefore, to assign to it a probability distribution and to calculate the probability that it lies between given values. In fact, it is often possible to calculate upper and lower confidence limits such that the probability that the interval between them includes the true value of θ is a specified number, say 0.95. This probability has a frequency interpretation, since if we continue for a long time taking sets of observations similar to the original set (always from the same population) and for each set calculate a 95 percent confidence interval, then in about 95 percent of cases the calculated confidence interval will include the true unknown value of θ. Fisher argues that this is academic nonsense. We do not in actual practice go on making hundreds of similar sets of observations and computing confidence intervals for each of them. We have *one* set, and on the basis of this set we want to make a reasonable statement about θ. Probability, being simply a measure of rational belief in a proposition, does not need a frequency interpretation.

In the binomial example quoted above, Neyman and Pearson would calculate symmetrical 95 percent confidence limits corresponding to $x = 40$ and $N = 100$ as 0.303 and 0.503, agreeing with Fisher's fiducial limits, but the interpretation would be different. They would say that this interval *includes* the true value with probability 0.95, while Fisher would say that 0.95 is the probability (subjective) that θ lies between these limits. On Neyman's objective theory, probabilities can be stated only for objects belonging to a fundamental probability set, a certain proportion of which have the property we are interested in. The fundamental probability set in the above example is the set of all confidence intervals calculated

for all the possible sets of N binomial trials, and the particular property is that of including, or covering, the true value θ. Whether we actually compute many confidence intervals or not, the concept is necessary to the frequency interpretation of probability.

In this view, a probability statement about θ has no meaning because it cannot be verified by noting the relative frequency of cases for which it is true. Nature has provided just one value of θ and not a hypothetical set of all possible values. Fisher has replied that the aggregate of all values of T (including the observed value), for which the inequality $T \leq K(\theta)$ is satisfied with relative frequency k, *can* be sampled, and continued sampling will demonstrate the correctness of the relative frequency. However, this argument seems to ignore the objection that the probability statement actually made refers not to T but to θ.

Fisher maintained that in the binomial situation mentioned above, we want to say something about θ on the basis of the one trial we have made, which has given a relative frequency of 0.4. The ratio of the likelihood of this sample to the maximum likelihood is given by

$$\frac{L}{L_{max}} = \frac{100^{100}}{40^{40}\ 60^{60}} \theta^{40} (1-\theta)^{60} \tag{6}$$

which measures the relative frequency with which the observed x (in this case 40) would have been produced by different values of θ. For $\theta = 0.30$, this ratio is 0.1045, and for $\theta = 0.50$ it is 0.1335, compared with 1 for $\theta = 0.40$. We could fix on an arbitrary small value of the likelihood ratio (say 1/10), calculate the two corresponding values of θ, and state that θ is very likely to lie within these limits. Fisher believed that limits so defined are more meaningful than the usual confidence limits of Neyman and Pearson.

In the well-known 'Student' t-test, the estimation of the population mean μ from a sample of size N with mean $\bar{x}$ and variance s^2 is the same on either theory of inference. If t is defined as $\sqrt{N}\,(\bar{x}-\mu)/s$, and if the population is assumed normal, the distribution of t is independent of σ^2, the population variance, as well as of μ. Thus, for $N = 10$, there is a probability 0.90 that t will lie between $\pm$ 1.812. The usual 90 percent confidence limits for μ are therefore given by

$$\mu = \bar{x} \pm 1.812\ s/\sqrt{10} \tag{7}$$

and Fisher would give the same values for fiducial limits. It is fortunate that in many common situations the two philosophies of inference lead to identical practical results. However, this is not always true, and one important exception is the Behrens-Fisher problem dealing with the difference of means between two populations with different variances.

Suppose $\bar{x}_1$ and $s_1{}^2$ are the sample mean and variance for a sample of size N_1, from a normal population with mean μ_1 and variance $\sigma_1{}^2$, and let subscript 2 denote similar quantities for a second sample from a second normal population. Then

$$\mu_1 - \mu_2 = \bar{x}_1 - \bar{x}_2 + t_1\ s_1\ N_1{}^{-\frac{1}{2}} - t_2\ s_2\ N_2{}^{-\frac{1}{2}} \tag{8}$$

where t_1 and t_2 have the Student t-distribution with N_1-1 and N_2-1 degrees of freedom, respectively. The distribution of $t_1\ s_1\ N_1{}^{-\frac{1}{2}} - t_2\ s_2\ N_2{}^{-\frac{1}{2}}$ as a function of s_1 / s_2 was worked out by Fisher and tabulated by Sukhatme (Reference [5]), and leads to a fiducial statement about $\mu_1 - \mu_2$ appropriate to the observed values of $\bar{x}_1$, $\bar{x}_2$, s_1, and s_2. It is not true, however, that in repeated samplings from the two fixed populations the fiducial limits would include the true value of $\mu_1 - \mu_2$ in the stated fraction of cases. They are not therefore confidence limits. Fisher maintained that they express, better than confidence limits, one's actual state of mind regarding $\mu_1 - \mu_2$ in view of the observed data. Fisher's calculation

really relates to a sub-set of all possible pairs of samples, a sub-set in which the ratio s_1 / s_2 is the same as for the pair actually observed.

Welch (Reference [6]) has suggested a test which does satisfy the condition that the relative frequency of truth of the confidence statement is equal to a specified value in repeated samplings from the two populations. Like Fisher's test, this test depends on the observed ratio of s_1 / s_2. Tables computed by Aspin are given in Biometrika Tables, Vol. 1 (Reference [7]), and, in general, lead to different numerical results from Sukhatme's tables.

When we reject a hypothesis at a fixed significance level α we say to ourselves: 'Either a rare event (one with probability α) has occurred or the hypothesis is not true,' and we prefer the second alternative. The probability $1-\alpha$ can be regarded as a measure of our rational disbelief in the hypothesis *after* the observations have been made. Frequently, of course, the scientist has previous suspicions about the hypothesis and devises an experiment to test it. He runs a certain risk, measured by α, of rejecting a true hypothesis, but he is prepared to take this risk. He would also like to make the chance of accepting a false hypothesis as small as possible and this means using a test with maximum power. Fisher seemed curiously reluctant to accept the idea of the power of a test. He says: 'To a practical man, also, who rejects a hypothesis, it is of course a matter of indifference with what probability he might be led to accept the hypothesis falsely, for in his case he is not accepting it.' It does seem, though, that the practical man ought to be interested in the extent to which the test he is proposing to use might lead him astray, and this is what the power tells him.

A more sophisticated treatment of fiducial inference has recently been given by John Tukey in the 1958 Wald lectures and by D. A. S. Fraser (Reference [8]). Fraser has shown, among other things, that one need not reject prior information, as Fisher seemed to require. Such information can be incorporated, as was done by Bayes long ago, to give a posterior or conditional fiducial probability.

Let T be a statistic represented by a point in a sample space $\mathcal{T}$, and θ a parameter represented by a point in a parameter space Ω, and let T be a minimal sufficient statistic for θ. Fisher always required that his fiducial inference should be based on a sufficient statistic; that is, one which gives essentially all the information regarding θ that could be obtained from the original observations X. In other words, the conditional distribution of X, given that $T = t$, is independent of θ. If the statistic T is also minimal, the data cannot be reduced further without losing sufficiency. As an example, if N items are observed from a normal population with mean μ and variance σ^2, T can be taken as the pair $(\bar{x}, s)$ and θ as the pair (μ, σ), where $\bar{x} = \Sigma x_i / N$, and $s^2 = \Sigma(x_i - \bar{x})^2 / (N-1)$. Of course, Σx_i and Σx_i^2 would also be sufficient and minimal. The spaces $\mathcal{T}$ and Ω are both upper half-planes in Euclidean two-dimensional space, since s and σ are never negative.

Fisher required in his fiducial method that all possible values of the parameter θ be equivalent, in the sense that the distribution of X should depend upon θ in exactly the same way whatever numerical value θ may have. Fraser has interpreted this requirement as an invariance under a group of transformations, and has shown that this assumption leads to a fiducial distribution for θ which may even be given a frequency interpretation in terms of frequencies for a special kind of repeated sampling. The following paragraphs attempt to summarize his views.

Let G be a group of one-one transformations of the space $\mathcal{T}$ into itself which is transitive and also isomorphic to the group H of transformations of Ω. That is, if θ is associated with T and $h\theta$ with gT, the mapping $G \leftrightarrow H$ is one-one and is an isomorphism.

A function of T and θ which has a fixed distribution independent of θ is called a *pivotal quantity*, and leads to a fiducial statement about θ. An example is Student's $t = \sqrt{N}\,(\bar{x}-\mu) / s$, the distribution of which depends only on N.

Let t_o be some fixed arbitrary reference point in $\mathcal{T}$ and let g_t be the element of G that carries t_o into t (a particular value of the random variable T), so that $g_t\, t_o = t$. Similarly, let h_θ be the element of H that carries a fixed θ_o into θ. We can then identify a point in the space $\mathcal{T}$ by the group element corresponding to it and similarly for points in Ω. In fact, because of the one-one relation, the group G could be used as either of the two spaces.

The variable $t = g_t\, t_o$ has a distribution with parameter θ. The inverse transformation h_θ^{-1} applied to t will produce a variable with a distribution having parameter θ_o, since h_θ^{-1} must obviously carry θ into θ_o. Therefore, $h_\theta^{-1}\, g_t\, t_o$ is a variable with a fixed distribution identified by θ_o. If we denote $h_\theta^{-1}\, g_t$ by g, then g is clearly an element of G which produces a variable with a fixed distribution when applied to the fixed point t_o in $\mathcal{T}$. It is a function of the sufficient statistic T and of the parameter θ and therefore is pivotal. The particular choice of t_o and θ_o makes no essential difference to g.

In the example of the normal population, we may take t_o and θ_o both as (0, 1). This of course implies standardization of the population and of the sample to mean 0 and variance 1. The transformation which carries a point (x, y) into (a+bx, by) may be denoted by [a, b]. This is equivalent to changing the origin of x to -a and the units of both x and y to 1/b. The transformation which takes (0, 1) into (μ, σ) is therefore $h_\theta = [\mu, \sigma]$ and that which takes (0, 1) into $(\bar{x}, s)$ is $g_t = [\bar{x}, s]$. The inverse transformation h_θ^{-1} is $\left[-\frac{\mu}{\sigma}, \frac{1}{\sigma}\right]$, since this will take (μ, σ) into (0, 1). We have then for the pivotal quantity g,

$$g = \left[-\frac{\mu}{\sigma}, \frac{1}{\sigma}\right] \quad \left[\bar{x}, s\right] = \left[\frac{\bar{x}-\mu}{\sigma}, \frac{s}{\sigma}\right] \tag{9}$$

The distribution of g is well known, since $(\bar{x} - \mu)N^{1/2}/\sigma$ is a standardized normal variate z and $(N-1)s^2/\sigma^2$ is a χ^2-variate with N-1 degrees of freedom.

The pivotal equation

$$g = h_\theta^{-1}\, g_t \tag{10}$$

can be solved to give

$$g_t = h_\theta\, g \tag{11}$$

or equivalently

$$t = g_t\, t_o = h_\theta\, g\, t_o \tag{12}$$

The frequency distribution of g, applied to the fixed point t_o, generates a frequency distribution on $\mathcal{T}$, and when multiplied by h_θ generates the frequency distribution of T itself. A fiducial distribution, however, relates to the parameter θ. We can obtain such a distribution formally by solving (11) for θ.

Thus

$$h_\theta = g_t\, g^{-1} \tag{13}$$

and therefore

$$\hat{\theta} = h_\theta \cdot \theta_o = g_t\, g^{-1}\, \theta_o \tag{14}$$

where the symbol $\hat{\theta}$ is used to indicate the fiducial nature of the variable and distinguish it from the actual parameter value θ corresponding to the distribution of T. Equation (14) indicates that given the observed value t we can derive a frequency distribution of possible values of θ. It is a *posterior* distribution, conditional on the observed t, the only prior information being the *form* of the frequency distribution of T without the numerical value of the parameter.

In the above example, we may write (9) as

$$g = \left[\, z/N^{1/2} \;,\quad \chi/(N-1)^{1/2} \right]$$

so that

$$g^{-1}\,\theta_o = \left(-\frac{z}{\chi}\left(\frac{N-1}{N}\right)^{\frac{1}{2}} \quad , \qquad (N-1)^{\frac{1}{2}}/\chi \right)$$

Then

$$\theta = (\mu, \sigma) = [\bar{x},\ s] \left(-\frac{z}{\chi}\left(\frac{N-1}{N}\right)^{\frac{1}{2}} \quad , \qquad (N-1)^{\frac{1}{2}}/\chi \right) \tag{15}$$

This gives

$$\hat{\mu} = \bar{x} - \frac{sz}{\chi}\left(\frac{N-1}{N}\right)^{\frac{1}{2}} = \bar{x} - \frac{ts}{N^{\frac{1}{2}}} \tag{16}$$

$$\hat{\sigma} = (N-1)^{\frac{1}{2}}\ s/\chi \tag{17}$$

where t is Student's t with $N-1$ degrees of freedom. The fiducial distribution may be interpreted as a frequency distribution of possible values, *relevant to the particular observed values* $\bar{x}$ *and* s. This population of values of μ and σ is created artificially by repeated sampling from normal distributions and transforming each sample, by a shift of origin and scale, so as to make the sample mean equal to $\bar{x}$ and the sample standard deviation equal to s. The necessary transformation will carry each mean μ and standard deviation σ to a new value and the set of these new values is the population which has the fiducial distribution.

With reference to the Behrens-Fisher problem, Fraser points out that the fiducial variable given by (8) gives a frequency distribution for $\mu_1 - \mu_2$ appropriate to the observed values of $\bar{x}_1$, $\bar{x}_2$, s_1, and s_2. However, a fiducial interval for

$\mu_1 - \mu_2$ is not invariant under transformations of x_1 and x_2 of the type considered above. The solution does not give frequencies consistent with those which would be obtained by repeated sampling from fixed populations, but perhaps the fixed population point of view is inappropriate for this problem. It seems that in discussions of the philosophy of statistical inference there is still much room for matters of taste.

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ON CHARACTERIZING NONADDITIVITY

Mary D. Lum

Summary

Since the validity of analysis of variance tests is strongly dependent on assumptions concerning additivity of effects, the question of how to characterize departures from additivity of main effects becomes an important consideration. If this can be done in a straightforward manner, it could conceivably indicate how the effects in a linear model may be made 'more additive' (that is, with fewer or no interactions) by a suitable transformation to a different scale. In this paper a multiple regression is proposed as a general basis for characterizing this nonadditivity. Numerical comparisons are made with the Tukey and Harter-Lum techniques.

1. Introduction

The assumptions underlying the analysis of variance tests are normality, homogeneity of variance, zero covariance, and additivity of effects. Early writers on the role of these assumptions, on the consequences when they are violated, and on the use of transformations to make the analysis more valid, included Fisher (Reference [8], Irwin (Reference [11]), Pearson (Reference [13]), Curtiss (Reference [5]), Eisenhart (Reference [6]), Cochran (Reference [4]), and Bartlett (Reference [2]). These papers, for the most part, dealt with adjustments for nonnormality and/or nonhomogeneity of variance. However, Box and Andersen (Reference [3]) have since indicated that these departures from the assumptions have no drastic effect on the overall analysis. Later writers who placed more emphasis on nonadditivity included Anscombe and Tukey (Reference [1]), Elston (Reference [7]), and Mandel (Reference [12]).

Indicating much concern about nonadditivity of main effects, Tukey (Reference [14]) proposed a test for a two-factor experiment in which a sum of squares for nonadditivity with a single degree of freedom is separated from the interaction sum of squares and tested against the residual. Harter and Lum (Reference [9]) interpreted

this separated sum of squares as that for linear regression of the sample interaction on the product of the sample main effects. They further showed (Reference [10]) that the linear regression of the sample two-factor (three-factor) interaction on the product of the population main effects is identical with its most general quadratic (cubic) regression on the population main effects. Based on these ideas they extended the test to experiments with three or more factors, specifying several indicators for testing nonadditivity, where each indicator corresponds to a linear regression of a sample interaction on the product of the corresponding sample main effects.

This paper describes a procedure for characterizing and detecting departures from additivity of main effects which is more general than the two above-mentioned procedures. The reasons why it is more general are: (1) it takes into account higher order nonadditivity which the Tukey procedure does not and which the Harter-Lum procedure does in a limited sense only; (2) more degrees of freedom are made available for testing nonadditivity than in either of the other two methods; and (3) one can make a more accurate diagnosis (indicating which variables are the primary causes for the nonadditivity) than with either of the other two methods. While only the case of a three-factor cross-classification design is discussed, the procedure can be extended to n factors and to incompletely replicated designs.

What is considered here is a general characterization of nonadditivity by a multiple regression with one or more degrees of freedom associated with it. This multiple regression of nonadditivity is one where the dependent variable represents deviations of observed cell means from the fitted linear model (the grand mean plus the sum of the observed main effects) and where the independent variables are chosen using the set of all possible products among observed main effects and all possible products (with the exception of the highest order interaction) of observed interactions with observed main effects. The mean square for this multiple regression is then a mean square for nonadditivity. If F represents the ratio of the mean square

for multiple regression to the residual mean square for the dependent variable defined above, then the particular independent variables chosen (to characterize nonadditivity) are those which yield the most significant F, F_s. Where F_s is significant at some preassigned level the corresponding independent variables are then significant contributors to nonadditivity.

2. Representing Departure from Additivity

The reasoning which led to consideration of a multiple regression for characterizing nonadditivity is given in the following discussion. We shall confine our discussion to a cross-classification design only. In the general linear model for a three-factor cross-classification experiment, the observed value x is represented as the sum of a grand mean M, the main effects A, B, C, the two-factor interactions AB, AC, BC, the three-factor interaction ABC, and a random variable e. In general, e must be small relative to the main effects and the interactions (a not unreasonable requirement) before one can evaluate nonadditivity with any precision.

Now x also happens to be identical with the sum of an estimated grand mean $\hat{M}$, estimated main effects $\hat{A}$, $\hat{B}$, $\hat{C}$, estimated two-factor interactions $\widehat{AB}$, $\widehat{AC}$, $\widehat{BC}$, estimated three-factor interaction $\widehat{ABC}$, and estimated error $\hat{e}$; the values with the circumflex are the usual analysis of variance estimates. The population values are not directly observable, but the corresponding estimates (with the circumflex) are calculable. It then follows that the multiple regression can only be taken on the estimates, the calculable quantities.

Practically speaking, nonadditivity of main effects implies that at least one of the estimated two-factor interactions and/or the estimated three-factor interaction is significant: otherwise, nonadditivity would be indistinguishable from e and any further discussion becomes academic. A significant two-factor interaction represents first-order departure from additivity of main effects. A significant three-factor

interaction represents second-order departure from additivity of main effects, over and above that accounted for by any significant two-factor interaction(s).

A significant $\widehat{AB}$ interaction implies that a significant portion of x must be a nonlinear function of $\hat{A}$ and $\hat{B}$ only, provided neither $\hat{A}$ nor $\hat{B}$ is identically zero. Similar remarks may be made for the $\widehat{AC}$ and $\widehat{BC}$ interactions. If more than one of the two-factor interactions is significant, a significant portion of x must be a nonlinear function of $\hat{A}$, $\hat{B}$, $\hat{C}$, provided two of the estimated main effects are not identically zero. Thus, the sum of the two-factor interactions has a portion which can be represented by a nonlinear function of $\hat{A}$, $\hat{B}$, $\hat{C}$. If this nonlinear function can be expressed as a power series or a polynomial, it must be of degree higher than one since it represents first-order departure from additivity of main effects. The lowest degree for a polynomial satisfying this requirement is two.

The terms of degree two are $\hat{A}^2$, $\hat{B}^2$, $\hat{C}^2$, $\hat{A} \times \hat{B}$, $\hat{A} \times \hat{C}$, $\hat{B} \times \hat{C}$. Now, the sum over the range of a subscript of an estimated effect is zero. Also, $\widehat{AB}_{i.} + \widehat{AB}_{.j} - \widehat{AB}_{..} = 0$, $\widehat{AC}_{i.} + \widehat{AC}_{.k} - \widehat{AC}_{..} = 0$, $\widehat{BC}_{j.} + \widehat{BC}_{.k} - \widehat{BC}_{..} = 0$, the dot notation indicating average. Because of these relations the terms $\hat{A}^2$, $\hat{B}^2$, $\hat{C}^2$ and the linear terms $\hat{A}$, $\hat{B}$, $\hat{C}$ drop out, and the most general quadratic representation of $\widehat{AB} + \widehat{AC} + \widehat{BC}$ by a polynomial in $\hat{A}$, $\hat{B}$, $\hat{C}$ is given by

$$b_1 (\hat{A} \times \hat{B}) + b_2 (\hat{A} \times \hat{C}) + b_3 (\hat{B} \times \hat{C})$$

where the '$\times$' symbol represents multiplication, and where b_1, b_2, b_3 are unknown constants to be estimated from the data. Generally speaking, this second-degree polynomial expression is the lowest possible for representing first-order nonadditivity. The independent variables $\hat{A} \times \hat{B}$, $\hat{A} \times \hat{C}$, $\hat{B} \times \hat{C}$ will be referred to as the 'first-order contributors to nonadditivity,' or simply 'first-order nonadditivities.'

Even though x is a linear function of $\hat{A}$, $\hat{B}$, $\hat{C}$, $\widehat{AB}$, $\widehat{AC}$, $\widehat{BC}$, $\widehat{ABC}$, a significant $\widehat{ABC}$ interaction implies that a significant portion of x must be a nonlinear

function of $\hat{A}$, $\hat{B}$, $\hat{C}$, $\widehat{AB}$, $\widehat{AC}$, $\widehat{BC}$ provided two of the estimated effects are not identically zero. If this nonlinear function can be expressed as a power series or a polynomial, it must be of degree higher than two since it represents second-order nonadditivity over and above first-order nonadditivity; otherwise it would vanish and make no contribution to second-order nonadditivity. The lowest degree for a polynomial satisfying this requirement is three. Also, since significant two-factor interactions are first-order nonadditivities, they are equivalent to at least a second-degree polynomial expression; otherwise they would vanish and make no contribution to first-order nonadditivity. The terms of degree three are $\hat{A}^3$, $\hat{B}^3$, $\hat{C}^3$, $\hat{A}^2 \times \hat{B}$, $\hat{A}^2 \times \hat{C}$, $\hat{A} \times \hat{B}^2$, $\hat{A} \times \hat{C}^2$, $\hat{B}^2 \times \hat{C}$, $\hat{B} \times \hat{C}^2$, $\hat{A} \times \hat{B} \times \hat{C}$, $\hat{A} \times \widehat{AB}$, $\hat{A} \times \widehat{AC}$, $\hat{A} \times \widehat{BC}$, $\hat{B} \times \widehat{AB}$, $\hat{B} \times \widehat{AC}$, $\hat{B} \times \widehat{BC}$, $\hat{C} \times \widehat{AB}$, $\hat{C} \times \widehat{AC}$, $\hat{C} \times \widehat{BC}$.

Again the sum over the range of a subscript of an estimated effect is zero. Also, $\widehat{ABC}_{ij.} + \widehat{ABC}_{i.k} + \widehat{ABC}_{.jk} - \widehat{ABC}_{i..} - \widehat{ABC}_{.j.} - \widehat{ABC}_{..k} + \widehat{ABC}_{...} = 0$. Because of these relations the terms $\hat{A}^3$, $\hat{B}^3$, $\hat{C}^3$, $\hat{A}^2 \times \hat{B}$, $\hat{A}^2 \times \hat{C}$, $\hat{A} \times \hat{B}^2$, $\hat{A} \times \hat{C}^2$, $\hat{B}^2 \times \hat{C}$, $\hat{B} \times \hat{C}^2$, $\hat{A} \times \widehat{AB}$, $\hat{A} \times \widehat{AC}$, $\hat{B} \times \widehat{AB}$, $\hat{B} \times \widehat{BC}$, $\hat{C} \times \widehat{AC}$, $\hat{C} \times \widehat{BC}$ and other terms of degree less than three drop out. We are left with

$$b_4\ (\hat{A} \times \hat{B} \times \hat{C}) + b_5\ (\hat{A} \times \widehat{BC}) + b_6\ (\hat{B} \times \widehat{AC}) + b_7\ (\hat{C} \times \widehat{AB})$$

as the third-degree (lowest possible) polynomial representation for second-order nonadditivity. The unknown constants b_4, b_5, b_6, b_7 are to be estimated from the data. The independent variables $\hat{A} \times \hat{B} \times \hat{C}$, $\hat{A} \times \widehat{BC}$, $\hat{B} \times \widehat{AC}$, $\hat{C} \times \widehat{AB}$ will be referred to as the 'second-order contributors to nonadditivity,' or simply 'second-order nonadditivities.'

Now $\hat{d}$, the estimated total departure from an additive main-effects model, consists of the sum of the estimated two-factor interactions and the estimated three-factor interaction. Therefore, the sum of the two polynomial expressions for first- and second-order nonadditivities represents the general polynomial

representation for $\hat{d}$ with the least number of terms, namely:

$$b_1 (\hat{A} \times \hat{B}) + b_2 (\hat{A} \times \hat{C}) + b_3 (\hat{B} \times \hat{C}) + b_4 (\hat{A} \times \hat{B} \times \hat{C}) + b_5 (\hat{A} \times \widehat{BC}) + b_6 (\hat{B} \times \widehat{AC}) + b_7 (\hat{C} \times \widehat{AB}) \tag{2.1}$$

In the case of a fixed factor model the symbols with circumflexes are unbiased estimates of the corresponding population values, and we use an identical argument to show that d, the total departure from an additive main-effects model, can be represented by the following polynomial expression:

$$b_1 (A \times B) + b_2 (A \times C) + b_3 (B \times C) + b_4 (A \times B \times C) + b_5 (A \times BC) + b_6 (B \times AC) + b_7 (C \times AB) \tag{2.2}$$

The only additional restriction needed to show (2.2) is that the sum over the range of a subscript of a population effect is zero (the usual restriction for fixed factors); see for example Reference [10], page 52.

It is important to note that the formal nature of (2.1) does not depend on whether the population effects are fixed or random. In other words, one can use (2.1) as a multiple regression to represent nonadditivity for random and mixed models, as well as for a fixed model.

To see how the method works, consider the following example of a three-factor experiment taken from Reference [10], page 14, on testing the strength of packing boxes. The data are the number of drops from a fixed height corresponding to three box styles, two load types, and two weights. Eight observations were made for each of the twelve combinations of levels of the three factors. The original data and the analysis of variance are given in Tables 2.1 and 2.2. Note that except for the $\widehat{SW}$ interaction all the effects are highly significant. To make the calculations

simpler, we first multiply each entry in Table 2.1 by 0.96. Neither the corresponding analysis of variance F-ratios nor the value of the multiple correlation coefficient is affected. However, the regression coefficients and the sum of squares for nonadditivity and residual will be changed slightly.

Table 2.1

Number of Drops of Failure of Packing Boxes Dropped from Fixed Height

Style of Box	1		2		4	
Load Type	1	3	1	3	1	3
	3	23	8	15	6	13
	4	30	8	13	5	13
	3	34	10	11	7	13
Weight	2	31	6	15	7	18
3	3	27	8	11	8	13
	4	30	7	9	8	18
	4	31	6	11	7	18
	3	27	6	12†	7†	15†
	26	233	59	97	55	121
	4	25	8	8	5	9
	7	15	6	11	6	11
	4	14	9	10	7	12
	6	26	9	9	7	15
Weight	6	29	5	8	6	12
4	5	22	6	9	5	9
	4	23	5	10	5	6
	9	23	6	12	7	16
	45	177	54	77	48	90

†Three missing observations; estimates supplied are cell means.

Table 2.2

Analysis of Variance

Source	d. f	S. S.	M. S.	F
Styles (S)	2	690.1458	345.0729	55.62**
Load Types (L)	1	2688.1667	2688.1667	433.32**
Weights (W)	1	104.1667	104.1667	16.79**
SL	2	1383.8958	691.9479	111.54**
SW	2	3.2708	1.6354	0.26
LW	1	135.3750	135.3750	21.82**
SLW	2	65.4375	32.7188	5.27**
Error	81††	502.5000	6.2037	
Total	92††	5572.9583		

**Significant at the 1 percent level.

††Three degrees of freedom lost because of missing observations.

The estimated or sample main effects and interactions, calculated in the usual manner, are given in Table 2.3 with the circumflex being used to denote 'estimated' effects. The corresponding population main effects and interactions will be given without the circumflex. For example

$$\hat{S}_1 = 0.96\left[\frac{26 + 233 + 45 + 177}{32} - \frac{1082}{96}\right] = 3.61$$

and S_1 represents the corresponding population value for Style 1.

Table 2.3

Estimated Main Effects

$\hat{S}_1 = 3.61$ $\hat{L}_1 = -5.08$ $\hat{W}_3 = 1.00$

$\hat{S}_2 = -2.21$ $\hat{L}_3 = 5.08$ $\hat{W}_4 = -1.00$

$\hat{S}_3 = -1.40$

Estimated Interaction Effects

$\widehat{SL}$

L	S = 1	S = 2	S = 4
1	-5.09	3.25	1.84
3	5.09	-3.25	-1.84

$\widehat{SW}$

W	S = 1	S = 2	S = 4
3	.11	-.25	.14
4	-.11	.25	-.14

$\widehat{LW}$

W	L = 1	L = 3
3	-1.14	1.14
4	1.14	-1.14

$\widehat{SLW}$

L	W	S = 1	S = 2	S = 4
1	3	-1.11	.69	.42
	4	1.11	-.69	-.42
3	3	1.11	-.69	-.42
	4	-1.11	.69	.42

As a consequence of the above discussion, we shall fit to the $\hat{d}$'s the regression (2.1), or that based on a suitable sub-set of the seven 'independent' variables of (2.1), by the usual method of least squares, using the estimates $\hat{S}$, $\hat{L}$, $\hat{W}$, $\widehat{SL}$, $\widehat{SW}$, $\widehat{LW}$ to calculate values of the independent variables. Accordingly, we minimize the sum of squared deviations from the fitted multiple regression. The departure $\hat{d}$ is obtained as the difference between the average cell value and the fitted additive

model. For example

$$\hat{d}_1 = 0.96\left(\frac{26}{8}\right) - \left[0.96\left(\frac{1082}{96}\right) + \hat{S}_1 + \hat{L}_1 + \hat{W}_1\right]$$

$$= 3.12 - 10.82 - 3.61 - (-5.08) - 1.00 = -7.23$$

Table 2.4 gives values of $\hat{d}$ (dependent variable) with corresponding values for $x_1, \ldots, x_7$ ('independent' variables) where $x_1 = \hat{S} \times \hat{L}$, $x_2 = \hat{S} \times \hat{W}$, $x_3 = \hat{L} \times \hat{W}$, $x_4 = \hat{S} \times \hat{L} \times \hat{W}$, $x_5 = \hat{S} \times \widehat{LW}$, $x_6 = \hat{L} \times \widehat{SW}$, $x_7 = \hat{W} \times \widehat{SL}$.

Table 2.4

Values for $\hat{d}$, x_1, x_2, x_3, x_4, x_5, x_6, x_7

α	ijk	$\hat{d}_\alpha$	$x_{1\alpha}$ $(\hat{S} \times \hat{L})$	$x_{2\alpha}$ $(\hat{S} \times \hat{W})$	$x_{3\alpha}$ $(\hat{L} \times \hat{W})$	$x_{4\alpha}$ $(\hat{S} \times \hat{L} \times \hat{W})$	$x_{5\alpha}$ $(\hat{S} \times \widehat{LW})$	$x_{6\alpha}$ $(\hat{L} \times \widehat{SW})$	$x_{7\alpha}$ $(\hat{W} \times \widehat{SL})$
1	113	-7.23	-18.3388	3.61	-5.08	-18.3388	-4.1154	-0.5588	-5.09
2	133	7.45	18.3388	3.61	5.08	18.3388	4.1154	0.5588	5.09
3	213	2.55	11.2268	-2.21	-5.08	11.2268	2.5194	1.2700	3.25
4	233	-3.05	-11.2268	-2.21	5.08	-11.2268	-2.5194	-1.2700	-3.25
5	413	1.26	7.1120	-1.40	-5.08	7.1120	1.5960	-0.7112	1.84
6	433	-0.98	-7.1120	-1.40	5.08	-7.1120	-1.5960	0.7112	-1.84
7	114	-2.95	-18.3388	-3.61	5.08	18.3388	4.1154	0.5588	5.09
8	134	2.73	18.3388	-3.61	-5.08	-18.3388	-4.1154	-0.5588	-5.09
9	214	3.95	11.2268	2.21	5.08	-11.2268	-2.5194	-1.2700	-3.25
10	234	-3.45	-11.2268	2.21	-5.08	11.2268	2.5194	1.2700	3.25
11	414	2.42	7.1120	1.40	5.08	-7.1120	-1.5960	0.7112	-1.84
12	434	-2.70	-7.1120	1.40	-5.08	7.1120	1.5960	-0.7112	1.84

3. Computation of Mean Squares for Nonadditivity

The resulting set of 'normal' equations yields the best linear unbiased estimates for the regression coefficients, $\hat{b}_1, \ldots, \hat{b}_7$. If $\underline{x}$ denotes the 12×7 information matrix, these equations can be expressed in matrix form* as:

$$12\, \underline{x}'\, \underline{\hat{d}} = \underline{S}\, \underline{\hat{b}}$$

where the prime denotes the transpose matrix,

$$\underline{S} = 12\, \underline{x}'\, \underline{x} = \left[12 \sum_{\alpha=1}^{12} x_{u\alpha}\, x_{v\alpha} \right] =$$

$$\begin{bmatrix} 24620.79 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 954.0576 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3716.122 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24620.79 & 5525.138 & 933.4897 & 6860.048 \\ 0 & 0 & 0 & 5525.138 & 1239.893 & 209.4839 & 1539.460 \\ 0 & 0 & 0 & 933.4897 & 209.4839 & 116.6862 & 271.8328 \\ 0 & 0 & 0 & 6860.048 & 1539.460 & 271.8328 & 1913.098 \end{bmatrix} \quad (3.1)$$

$$\underline{\hat{b}} = \begin{bmatrix} \hat{b}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{b}_7 \end{bmatrix} \qquad 12\, \underline{x}'\, \underline{\hat{d}} = \left[12 \sum_{\alpha=1}^{12} \hat{d}_\alpha\, x_{u\alpha} \right] = \begin{bmatrix} 6860.048 \\ 36.17284 \\ 833.9328 \\ 1492.3008 \\ 334.8864 \\ 57.49747 \\ 415.9296 \end{bmatrix}$$

$$u, v = 1, \ldots, 7$$

* The multiplication by 12 (the total number of cells) was solely to facilitate computation using a general Fortran program for the IBM 1620 Electronic Computer.

The reasons for exhibiting the normal equations in matrix form are to show the nature and arrangement of the coefficients for the $\hat{b}$'s and to adapt these equations for computation on the IBM 1620 Electronic Computer. The augmented matrix $\underline{r}$ of computed paired correlations is obtained as

$$\underline{r} = [r_{uv}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & .550743 & .999557 \\ 0 & 0 & 0 & 1 & 1 & .550743 & .999557 \\ 0 & 0 & 0 & .550743 & .550743 & 1 & .575339 \\ 0 & 0 & 0 & .999557 & .999557 & .575339 & 1 \\ .933119 & .0249951 & .291976 & .202986 & .202986 & .113606 & .202961 \end{bmatrix} \tag{3.2}$$

where the eighth row gives the paired correlations of $\hat{d}$ with $x_1, x_2, \ldots, x_7$, respectively.

In connection with (3.2) some general comments can be made which are pertinent to the subsequent discussion:

(1) The zero correlations arise because the estimated effects sum to zero;

(2) x_1 is uncorrelated with the rest of the independent variables; the same is true of x_2, x_3;

(3) x_1, x_2, x_3, x_4 are uncorrelated among themselves; thus if one is interested in regression on x_1, x_2, x_3, x_4 only (or any subgroup of these four variables), one has the very convenient property that the multiple regression of $\hat{d}$ on x_1, x_2, x_3, x_4 is equivalent to the sum of the four individual regressions on each of the x's.

For this example, it is important to point out that observations were made for only two levels each of Load Type and Weight. Hence, the sum-to-zero restriction forces $\widehat{LW}_{jk}$ to be proportional to $\hat{L}_j \times \hat{W}_k$ $(j,k = 1,2)$, and x_5 is linearly dependent on x_4. That this is so can be verified by noting that Columns 4 and 5 of (3.2) are identical or that the ratio of a number in Column 5 of (3.1) to the number in Column 4 (of the same row) does not change with rows. Because of the dependence of x_5 on x_4 for this example, we must discard x_5 in order to obtain a non-singular matrix $\underline{S}$. Otherwise, a unique solution for the regression coefficients is not possible. Accordingly, we deleted Row 5 and Column 5 of (3.1) before solving for the $\hat{b}$'s.

Values for the partial correlation coefficients are given in Table 3.1.

Table 3.1

Partial Correlation Coefficients*

(1) $r_{uv.p} = r_{uv.pq} = r_{uv.pqr} = r_{uv.pqrs} = 0$ where $u = 1, 2, 3$; $v > u$; and where p, q, r, s represent any other subscripts except 5 and 7 of the independent variables different from u and v and different from one another.

(2) $r_{uv.p} = r_{uv.pq} = r_{uv.pqr} = .55074$ for $u = 4$, $v = 6$.

	$v = 1$	$v = 2$	$v = 3$	$v = 4$	$v = 6$
$r_{dv.1}^{**}$	-	.06951	.81202	.56453	.31595$^+$
$r_{dv.2}$	.93341	-	.29207	.20305	.11364
$r_{dv.3}$	.97563	.02613	-	.21223	.11878
$r_{dv.4}$	.95296	.02553	.29818	-	.00222

*The variables x_5 and x_7 which are highly correlated with x_4 are omitted here.

**The subscript d indicates the total deviation from additivity, $\hat{d}$.

Table 3.1 (Continued)

	v = 1	v = 2	v = 3	v = 4	v = 6
$r_{dv.6}$	.93920	.02516	.29388	.16933	-
$r_{dv.12}$	-	-	.81399	.56590	.31672
$r_{dv.13}$	-	.11911	-	.96728	.54136
$r_{dv.14}$	-	.08422	.98378	-	.00073
$r_{dv.16}$	-	.07327	.85586	.49313	-
$r_{dv.23}$	.97596	-	-	.21231	.11882
$r_{dv.24}$	.95327	-	.29828	-	.00222
$r_{dv.26}$	.93950	-	.29397	.16938	-
$r_{dv.34}$	.99838	.02674	-	-	.00232
$r_{dv.36}$	.98259	.02632	-	.17715	-
$r_{dv.46}$	.95296	.02553	.29818	-	-
$r_{dv.123}$	-	-	-	.97421	.54524
$r_{dv.124}$	-	-	.98728	-	.00734
$r_{dv.126}$	-	-	.85817	.49446	-
$r_{dv.134}$	-	.46943	-	-	.04078
$r_{dv.136}$	-	.14166	-	.95346	-
$r_{dv.146}$	-	.08422	.98380	-	-
$r_{dv.234}$	.99873	-	-	-	.00232
$r_{dv.236}$	.98293	-	-	.17721	-
$r_{dv.346}$	.99838	.02674	-	-	-
$r_{dv.1234}$	-	-	-	-	.04619
$r_{dv.1236}$	-	-	-	.96317	-
$r_{dv.1246}$	-	-	.98731	-	-
$r_{dv.1346}$	-	.46982	-	-	-
$r_{dv.2346}$	.99874	-	-	-	-

Table 3.2 exhibits the computed regression coefficients for eleven fitted regressions, each with the associated value of its computed multiple correlation coefficient $\hat{R}$.

Table 3.2

Computed Regression Coefficients and Multiple Correlation

Regression	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{b}_5$*	$\hat{b}_6$	$\hat{b}_7$	$\hat{R}$
(1)	.278628	.037915$^-$	.224409	-.088140	-	-.067108	.543003	.9989035$^-$
(2)	.278628	.037915$^-$	.224409	.060184	-	.011284	-	.9988965$^-$
(3)	.278628	.037915$^-$	.224409	.060611	-	-	-	.9988942
(4)	.278628	-	.224409	.060611	-	-	-	.9985816
(5)	.278628	-	.224409	-	-	-	-	.9777329
(6)	.278628	.037915$^-$	.224409	-	-	-	-	.9780525$^+$
(7)	$\hat{b}_{123}$ = .263909			-				.9640110
(8)	$\hat{b}_{13}$ = .271518			-				.9755186
(9)	$\hat{b}_{123}$ = .263909			$\hat{b}_{4567}$ = .039280				.9851311
(10)	$\hat{b}_{13}$ = .271518			$\hat{b}_4$ = .060611				.9964135$^-$
(11)	$\hat{b}_{13}$ = .271518			$\hat{b}_{457}$ = .040327				.9964154

(1) Regression of $\hat{d}$'s on x_1, x_2, x_3, x_4, x_6, x_7

(2) Regression of $\hat{d}$'s on x_1, x_2, x_3, x_4, x_6

(3) Regression of $\hat{d}$'s on x_1, x_2, x_3, x_4

(4) Regression of $\hat{d}$'s on x_1, x_3, x_4 (equivalent to sum of three separate regressions in Harter-Lum procedure)

(5) Regression of $\hat{d}$'s on x_1, x_3

(6) Regression of $\hat{d}$'s on x_1, x_2, x_3

(7) Regression of $\hat{d}$'s on $x_1 + x_2 + x_3$ (equivalent to general procedure suggested by Tukey)

(8) Regression of $\hat{d}$'s on $x_1 + x_3$

(9) Regression of $\hat{d}$'s on $x_1 + x_2 + x_3$, $x_4 + x_5 + x_6 + x_7$

(10) Regression of $\hat{d}$'s on $x_1 + x_3$, x_4

(11) Regression of $\hat{d}$'s on $x_1 + x_3$, $x_4 + x_5 + x_7$

* x_5 is linearly dependent on x_4 in this example.

Table 3.3 exhibits the corresponding sum of squares for regression, sum of squares for residual, and F-value. For comparison purposes, $F_{n,r}(.01)$ is the tabulated value for a central F-distribution with n and r degrees of freedom at the .01 level of significance, P is the area under the right tail for a central F-distribution corresponding to the F-value given.

Table 3.3

Nonadditivity Sum of Squares and its F-ratio

Regr.	Regression (Nonadditivity) Sum of Squares	Residual Sum of Squares	n	r	F	$F_{n,r}(.01)$	P
(1)	1460.2738	3.2077	6	1	76	5859	>.05
(2)	1460.2534	3.2281	5	2	181	99	$.005^{+}$
(3)	1460.2466	3.2349	4	3	339	29	$.0005^{+}$
(4)	1459.3330	4.1485^{+}	3	4	469**	17	.00002
(5)	1399.0323	64.4492	2	5	54	13	.0004
(6)	1399.9470	63.5345^{-}	3	4	29	17	.004
(7)	1360.0385^{+}	103.4430	1	6	79	14	.00011
(8)	1392.7026	70.7789	1	6	118	14	.00004
(9)	1420.2842	43.1973	2	5	82	14	$.00015^{+}$
(10)	1453.0030	10.4785^{+}	2	5	347**	13	.000004
(11)	1453.0082	10.4733	2	5	347**	13	.000004

The figures in the column headed by n represent degrees of freedom for regression. The figures in the column headed by r represent degrees of freedom for residual.

**While the exact distribution of F is not known, the values are so large as to be regarded as significant indications of nonadditivity.

The sum of squares for the $\hat{d}$'s can be obtained by summing the sums of squares for the interactions and multiplying by $(0.96)^2$; that is, (sum of squares for d) $= (0.96)^2$ [1383.8958 + 3.2708 + 135.3750 + 65.4375] = 1463.4815$^+$.

Then, for example, the sum of squares for multiple regression for Regression (2) is $(.9988965)^2$ (1463.4815$^+$) = 1460.2534 and the sum of squares for its residual is 1463.4815$^+$ - 1460.2534 = 3.2281. The F-ratio corresponding to Regression (2) is given by:

$$\frac{1460.25}{5} \div \frac{3.2281}{2} = 181$$

Its corresponding P-value is obtained by utilizing the relationship between the Beta- and F-distribution (see Reference [16], page 187) and entering Karl Pearson's 'Tables of the Incomplete Beta Function,' (1934). Thus,

$$x = 1 - \frac{1}{1 + \frac{5}{2}(181)} = .997795^-$$

yields

$$I_x\left(\frac{5}{2}, \frac{2}{2}\right) = .99453 \quad \text{and} \quad P = 1 - I_x(2.5,\ 1) = .00547$$

The fitting of Regression (1) corresponds to the use of all independent variables indicated in (2.1), with the exception of x_5, which is linearly dependent on x_4. It should be noted that x_7 is also highly correlated with x_4; this gives a good reason for omitting x_7 and saving a degree of freedom for residual. Omitting x_7 yields Regression (2). Now since the $\widehat{SW}$ interaction was not significant it is not surprising that the x_6 term, $\hat{b}_6$ $(\hat{L} \times \widehat{SW})$, in the regression is small. Omitting x_6 results in Regression (3). Since the regression term in x_2, $\hat{b}_2$ $(\hat{S} \times \hat{W})$, is also small, we can further omit x_2, resulting in Regression (4) on x_1, x_3, x_4,

only. If we go even further and omit x_4, the result is Regression (5). Regression (6) on x_1, x_2, x_3 was computed for purposes of comparison with Tukey's proposed general test.

It is obvious that one cannot use up all the seven degrees of freedom available to the $\hat{d}$'s in order to fit the multiple regression to account for nonadditivity; otherwise no degrees of freedom will be left for the residual, and no test can be made to determine whether the x's are significant contributors to nonadditivity or not. Then the question arises as to how many of these degrees of freedom should be used for fitting a multiple regression to represent nonadditivity associated with appropriate x's. An apparently desirable goal would be to obtain as sensitive a test as possible, in which case one would naturally wish to obtain the most significant F; that is, the one corresponding to the maximum cumulative probability. We shall denote it by F_s. An examination of Table 3.3 indicates that among the regressions discussed so far (on individual x's) F is the most significant for Regression (4). One notes that Regression (4) will be a good fit to the d's since its corresponding R value, .99858, is only very slightly lower than the maximum R value listed, .99890.

The significance of F corresponding to Regression (4) indicates that the most important individual contributors to nonadditivity for this example, listed in order of importance (that is, according to magnitude of their contribution to the regression) are: $\hat{S} \times \hat{L}$, $\hat{L} \times \hat{W}$, $\hat{S} \times \hat{L} \times \hat{W}$. If a suitable transformation is to remove most of this nonadditivity it must be able to take these contributors into account. A good criterion for choosing an optimum transformation to reduce nonadditivity would seem to be: pick that transformation which minimizes the significance of F_s, the most significant F. It will be apparent in the next section that the most significant F in the example corresponds to Regression (10) on $x_1 + x_3$, x_4 and not to Regression (4) on x_1, x_3, x_4.

One further point should be noted here. The calculations for deriving (3) or (4) are extremely simple compared to those for the more general Regressions (1) and (2). The reason for this is that x_1, x_2, x_3, x_4 are uncorrelated among themselves. Thus $\underline{r}$ becomes a 4×4 identity matrix and the normal equations simplify to

$$24620.79\ \hat{b}_1 = 6860.048\ ,$$

$$954.0576\ \hat{b}_2 = 36.17284\ ,$$

$$3716.122\ \hat{b}_3 = 833.9328\ ,$$

$$24620.79\ \hat{b}_4 = 1492.3008\ ,$$

making it extremely easy to solve for $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$, $\hat{b}_4$.

Columns 3, 4, and 5 of Table 3.4 give the predicted d's using Regressions (2), (3), and (4). The predicted values agree quite well with the calculated $\hat{d}$'s given in Column 2. Table 3.4 appears on page 138.

4. Relationship to Harter-Lum and Tukey Methods

The Harter-Lum procedure separately fits $\hat{b}_1 x_1$ to the $\widehat{SL}$ interaction, fits $\hat{b}_2 x_2$ to the $\widehat{SW}$ interaction, fits $\hat{b}_3 x_3$ to the $\widehat{LW}$ interaction, and fits $\hat{b}_4 x_4$ to the $\widehat{SLW}$ interaction with one degree of freedom for nonadditivity associated with each fit. Table 4.1 lists the sum of squares for nonadditivity (that is, the sum of squares for regression) associated with each interaction. Two drawbacks of the Harter-Lum procedure are: (1) more than one indicator for nonadditivity; and (2) the strong dependence of the nonadditivity indicators on the separability of the interactions and then on the availability of degrees of freedom from at least one interaction to assign to residual. For the case of cross-classification designs which we are considering in this paper, these objections are easily circumvented by

Table 3.4

Predicted d's

α	Estimated d_α	d_α predicted by Regression (2)	Regression (3) (x_1, x_2, x_3, x_4)	Regression (4) (x_1, x_3, x_4)	Regression (7) $(x_1 + x_2 + x_3)$	Regression (8) $(x_1 + x_3)$	Regression (10) $(x_1 + x_3, x_4)$
1	-7.23	-7.22	-7.22	-7.36	-5.23	-6.36	-7.47
2	7.45	7.50	7.50	7.36	7.13	6.36	7.47
3	2.55	2.59	2.58	2.67	1.04	1.67	2.35^-
4	-3.05	-2.76	-2.75^+	-2.67	-2.21	-1.67	-2.35^-
5	1.26	1.21	1.22	1.27	0.17	0.55	0.98
6	-0.98	-1.31	-1.33	-1.27	-0.91	-0.55	-0.98
7	-2.95	-3.00	-3.00	-2.86	-4.45^+	-3.60	-2.49
8	2.73	2.72	2.72	2.86	2.55^-	3.60	2.49
9	3.95	3.66	3.67	3.59	4.89	4.43	3.75^-
10	-3.45	-3.49	-3.50	-3.59	-3.72	-4.43	-3.75^-
11	2.42	2.75	2.74	2.69	3.59	3.31	2.88
12	-2.70	-2.65^-	-2.64	-2.69	-2.85^-	-3.31	-2.88

considering a 'pooled' mean square. Because of (2), however, the Harter-Lum procedure cannot be applied to designs involving confounded interactions. It will be seen that the more general multiple regression approach advocated here does not suffer from such a limitation, since one can consider suitable sums of the independent variables in such situations.

Table 4.1

Sum of Squares

Interaction	Sum of Squares for Nonadditivity	Degrees of Freedom
$\widehat{SL}$	1382.6702	1
$\widehat{SW}$	0.9921	1
$\widehat{LW}$	135.3750	1
$\widehat{SLW}$	65.4300	1

With regard to a pooled mean square, we note that x_1, x_2, x_3, x_4 are uncorrelated. Therefore, the regression sum of squares for these four variables or any sub-set of them is equivalent to 'pooling' the mean squares; that is, summing the corresponding separate sums of squares for nonadditivity of the Harter-Lum procedure and associating the sum of the corresponding number of degrees of freedom with this pooled sum of squares. Thus, for example,

$$(0.96)^2\ [1382.670 + 135.375 + 65.430] = 1459.33$$

is the sum of squares for nonadditivity for a pooled Harter-Lum procedure with three associated degrees of freedom. One is then able to obtain a pooled mean square for nonadditivity from the Harter-Lum procedure which is equivalent to Regression (4) on x_1, x_3, x_4. Thus, this pooled test for nonadditivity is a special case of the more general multiple regression procedure.

On the other hand, for a cross-classification design the general test for nonadditivity suggested by Tukey (Reference [15]) is equivalent to that based on regression of the $\hat{d}$'s on $(x_1 + x_2 + x_3)$ with one degree of freedom associated with the regression. To show this equivalence, let k_{ijk} (following Tukey's notation) represent the fitted additive model. Then

$$k_{ijk} = \text{grand mean} + \hat{S}_i + \hat{L}_j + \hat{W}_k$$

and it follows that

$$y_{ijk} = (k_{ijk} - \bar{k}_{...})^2 = (\hat{S}_i + \hat{L}_j + \hat{W}_k)^2$$

$$= \hat{S}_i{}^2 + \hat{L}_j{}^2 + \hat{W}_k{}^2 + 2(\hat{S}_i \times \hat{L}_j) + 2(\hat{S}_i \times \hat{W}_k) + 2(\hat{L}_j \times \hat{W}_k)$$

Now

$$h_{ijk} = y_{...} + (y_{i..} - y_{...}) + (y_{.j.} - y_{...}) + (y_{..k} - y_{...}) = \hat{S}_i{}^2 + \hat{L}_j{}^2 + \hat{W}_k{}^2$$

Hence,

$$y_{ijk} - h_{ijk} = 2\left[(\hat{S}_i \times \hat{L}_j) + (\hat{S}_i \times \hat{W}_k) + (\hat{L}_j \times \hat{W}_k)\right] = 2(x_1 + x_2 + x_3)$$

Tukey's suggested general test for nonadditivity corresponds to regression of $\hat{d}_{ijk}$ on $(y_{ijk} - h_{ijk})$. Therefore, it is exactly the same as the test corresponding to regression of $\hat{d}_{ijk}$ on $(x_1 + x_2 + x_3)$.

The seventh row of Table 3.3 indicates that a Tukey regression (Regression (7)) on $x_1 + x_2 + x_3$ with one degree of freedom yielded a more significant indication of nonadditivity than Regression (6) on x_1, x_2, x_3 separately with three associated degrees of freedom. However, it is not nearly as significant an indication as the F-value of 469 associated with Regression (4). Also, it will be seen later

that the Tukey regression does not have as significant an indication of nonadditivity as the to-be-discussed Regressions (8), (10), and (11).

Nevertheless, at this point of the discussion, except for Regression (4) the regression on $(x_1 + x_2 + x_3)$ gave a better indication of nonadditivity than Regressions (1) to (6) on the individual x's. Economy in the use of degrees of freedom now looms up even more sharply as a factor to be contended with, and the question arises as to whether regression on some other sum or linear combination of sums of the x's might lead to a more significant F-value. An answer to this question may well be worthwhile for a situation where numbers of degrees of freedom for interactions are extremely limited, especially whenever such a situation leaves one no choice but to consider sums or a linear combination of sums of the x's. In this connection, five more regressions were tried. They were of the $\hat{d}$'s on:

(a) $b_{13}(x_1 + x_3)$, denoted by Regression (8);

(b) $b_{123}(x_1 + x_2 + x_3) + b_{4567}(x_4 + x_5 + x_6 + x_7)$, Regression (9);

(c) $b_{13}(x_1 + x_3) + b_{457}(x_4 + x_5 + x_7)$, Regression (10);

(d) $b_{13}(x_1 + x_3) + b_4 x_4$; Regression (11);

(e) $b_{134}(x_1 + x_3 + x_4)$.

Since $\hat{b}_2 x_2$ is comparatively small, Regression (8) omits x_2. Regression (9) adds the effect of the sum of the second-order nonadditivities to that of the sum of the first-order nonadditivities. Regression (10) is essentially the same as Regression (9) except that x_2 is omitted (since $\hat{b}_2 x_2$ is small compared to the regression terms in x_1 and x_3), and x_6 is omitted (since $\hat{b}_6 x_6$ is small compared to the terms in x_4, x_5, x_7). Regression (11) is essentially the same as Regression (10) except that x_5 and x_7 are omitted (being both highly correlated with x_4). The eighth through eleventh rows of Tables 3.2 and 3.3 give the regression coefficients, multiple correlation coefficient, sum of squares for regression and residual, and the F and P values corresponding to each of Regressions (8) through (11).

The regression represented by (e), on the sum of the three most important individual contributors to nonadditivity, yielded a comparatively low value of R (.84542), so no further calculations were made for this regression; consequently, it was not included in Tables 3.2 and 3.3. This poor showing verifies that one should not indiscriminately include second-order nonadditivities in a sum of first-order nonadditivities.

It is to be noted that Regression (8) on $x_1 + x_3$ yielded larger R and F values (R = .97552, F = 118) than those for the Tukey regression on $(x_1 + x_2 + x_3)$ with R = .96401, F = 79. This better result shows that the sum of a suitable sub-set of the first-order nonadditivities can yield a better indicator of non-additivity than the sum of *all* first-order nonadditivities. Regression (8), corresponding to the sum $x_1 + x_3$, yielded a much more significant indication of nonadditivity than Regression (5), on x_1 and x_3 separately. Actually this indication was about as significant as that for Regression (4) on x_1, x_3, x_4 separately; of course, it was much less significant than that for Regression (10) on $x_1 + x_3$, x_4.

Strangely enough, Regression (9), which one might have intuitively expected to be more sensitive to nonadditivity, did not show any improvement over the Tukey Regression (7) here.

Regression (10) yielded essentially the same values for R and F as Regression (11). This was as expected, since x_5 and x_7 are highly correlated with x_4, so that $x_4 + x_5 + x_7$ is almost equivalent to x_4. Regression (10) had much larger R and F values (R = .99641, F = 347) than Regression (9) (R = .98513, F = 82). In fact, the F-value (347) for Regression (10) is much more significant than that obtained (469) for Regression (4). In short, Regression (10) on $x_1 + x_3$, x_4, yielded the overall most significant F, F_s. This last result indicates that among the linear combinations of $x_1, x_2, \ldots, x_7$ characterizing nonadditivity,

the best representation may be the sum of a suitable sub-set of significant first-order nonadditivities and the sum of a suitable sub-set of significant second-order nonadditivities.

5. Conclusions

Where nonadditivity is present the evidence points to the following conclusions for a cross-classification experiment:

(1) A multiple regression approach is necessary for picking out the significant contributors to nonadditivity. Based on the most significant F for regression on individual x's, these are x_1 and x_3 (first-order nonadditivities) and x_4 (second-order nonadditivity) in the example considered.

(2) Multiple regressions involving only the significant contributors to nonadditivity yield the most sensitive indications of nonadditivity. If the sum of significant first-order nonadditivities and the significant second-order nonadditivity are used then $\hat{b}_{13}(x_1 + x_3) + \hat{b}_4 x_4$ yields the most significant F overall, F_s, for the example. Nearly optimal are $\hat{b}_1 x_1 + \hat{b}_3 x_3 + \hat{b}_4 x_4$ (equivalent to a pooled Harter-Lum procedure), associated with the most significant F for regression on individual x's, and $\hat{b}_{13}(x_1 + x_3)$. The latter yields, of course, the most significant F corresponding to one degree of freedom. It is a more sensitive indication in the example than a generalized test for nonadditivity suggested by Tukey (which is equivalent to using the sum of all first-order nonadditivities, $x_1 + x_2 + x_3$).

(3) Whether an effect is significant or relatively near zero (by the usual Analysis of Variance tests and estimates) has a strong association with which ones of $x_1, x_2, \ldots, x_7$ become the important contributors to nonadditivity. Table 2.2 indicates all effects are significant at the 1 percent level except for the $\widehat{SW}$ interaction which has an F less than one. Table 2.3 indicates that values of the estimated $\widehat{SW}$ interaction are nearly zero relative to the other effects. The

significance of the $\widehat{SLW}$ three-factor interaction along with the significance of $\hat{S}$, $\hat{L}$, $\hat{W}$ can be associated with x_4 $(\hat{S} \times \hat{L} \times \hat{W})$, the significant second-order contributor to nonadditivity; that $\widehat{SW}$ is nearly zero can be associated with the comparatively small regression coefficients for x_2 $(\hat{S} \times \hat{W})$ and x_6 $(\hat{L} \times \widehat{SW})$, making x_2 and x_6 insignificant contributors to nonadditivity. Of course, x_5 and x_7, being highly correlated with x_4, cannot be considered as contributors to nonadditivity because of their linear dependence on x_4. This leaves x_1 $(\hat{S} \times \hat{L})$, x_3 $(\hat{L} \times \hat{W})$, the two significant first-order contributors to nonadditivity which can be associated with the significant effects, $\widehat{SL}$, $\widehat{LW}$, $\hat{S}$, $\hat{L}$, $\hat{W}$.

The author advocates the above-discussed multiple regression procedure as a more general characterization of nonadditivity than that proposed by Tukey or by Harter and Lum. The greatest disadvantage of such a procedure appears to be the possible computational complexities necessitating use of a high-speed computer with its attendant annoyances and troubles. However, for the situation where it is known that x_1, x_2, x_3, x_4 (or a sub-set thereof) are the only significant contributors to nonadditivity the computations are extremely simple.

While only the cross-classification design is considered here, the author sees no obstacles to using a multiple regression approach in the case of designs with confounded interactions, such as latin squares for example. The only requirement is that at least one degree of freedom be available for nonadditivity and one degree of freedom (preferably a few more) for residual in order that a test for nonadditivity can be made. In this case one may have to resort to sums of the nonadditivity contributors such as those in Regressions (7) to (11).

In closing, it should be pointed out that further study needs to be made on the distribution of the F's, which are conjectured to be quite complicated, and on appropriate transformations to remove some of the nonadditivity. As suggested earlier, one possible criterion may be to choose a transformation which minimizes the maximum cumulative probability of the F's. Whether this is a feasible task

to accomplish or not remains to be answered by further research work along these lines.

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A NOTE ON THE STATISTICAL THEORY OF TURBULENCE

Knox Millsaps

1. Introduction

The mathematical essence of the analytical description (Reference [1]) for the behavior of homogeneous-isotropic turbulence within a fluid with no mean-mean motion is contained in the partial integro-differential equation:

$$\frac{\partial}{\partial t} \int_0^k F(x, t)\,dx = - \int_0^k T(y, t)\,dy - 2\nu \int_0^k z^2 F(z, t)\,dz \qquad (1.1)$$

wherein k is the wave number; that is, 2π times the number of turbulent waves per unit length, $[\ell^{-1}]$,

x, y, and z are dummy variables, $[\ell^{-1}]$;

t is the time, $[t]$;

ν is the kinematic viscosity, $[\ell^2\ t^{-1}]$;

$F(k, t)$ is the spectral function for the turbulent energy, $[\ell^3\ t^{-2}]$;

and $T(k, t)$ is the inertial transfer function for the turbulent energy, $[\ell^3\ t^{-3}]$.

One immediately notes that equation (1.1) has two unknown functions, $F(k, t)$ and $T(k, t)$, and that progress can apparently be made only after one postulates on intuitive physical grounds an additional relation between the two unknown functions. The apparent nature should be emphasized, for it is not inconceivable that someone may demonstrate the theoretical validity of another mathematical relation between the two unknown functions, either for the general theory or for a particular case. However, be the conjectures as they may, an intellectually appealing postulate has been given by von Kármán (Reference [2]), and it seems to be worthwhile to examine particular cases of the von Kármán suggestion in some detail.

2. The von Kármán Transfer Function

It appears to be appropriate to reproduce for the reader the von Kármán description of the turbulent cascade since a clear understanding of its physical foundation is required to appreciate the suggestion which is advanced in the present paper. The author's reproduction of the von Kármán masterwork is the following: In the turbulent spectrum $F(k, t)$ consider three wave number bands of widths dk_1, dk_2, and dk_3 which are located respectively at k_1, k_2, and k_3. Now, if $T(k_2, t)\, dk_2$ is considered to be the resultant turbulent energy entering the band dk_2 at k_2 per unit time per unit wave number (per unit mass, understood), then it should be regarded as the sum of energy contributions from every other wave band in the spectrum. Hence, it is logical to write a typical contribution to dk_2 at k_2 from dk_1 at k_1 as

$$dT(k_2, k_1, t) = \theta[F(k_2, t), F(k_1, t), k_2, k_1]\, dk_1$$

and, if $k_2 > k_1$, then $dT(k_2, k_1, t) > 0$ because of the implied assumption concerning the unidirectional cascading of the turbulent energy. Analogously, if $k_3 > k_2$, then an expression which typically represents the energy departing from dk_2 at k_2 for dk_3 at k_3 is

$$dT(k_3, k_2, t) = -\,\theta[F(k_3, t), F(k_2, t), k_3, k_2]\, dk_3$$

and the resultant transfer of energy at $k = k_2$, is

$$T(k, t) = \int_0^k \theta[F(k, t), F(x, t), k, x]\, dx - \int_k^\infty \theta[F(y, t), F(k, t), y, k]\, dy \tag{2.1}$$

where x and y are dummy variables. It is seemingly necessary to assign a mathematical form to θ, and the simplest form appears to be

$$\theta[F(k_1, t), F(k_2, t), k_1, k_2] = \gamma k_1^{m} k_2^{q} F^{n}(k_1, t) F^{p}(k_2, t) \qquad (2.2)$$

where γ is an absolute constant, [0]. A serious criticism can be made of this simplification since the structure of the Fourier transforms of the Reynolds' average of the Navier-Stokes equations suggests in a very strong manner that $F(k_1 - k_2, t)$ should be explicitly displayed. However, if it were introduced, the separation of the integration ranges would no longer be possible; consequently, with the engineering simplicity that is a characteristically personal attribute von Kármán chose the more tractable form.

After substituting equation (2.2) into equation (2.1) and after substituting the results of the earlier substitution into equation (1.1), one then notes from purely dimensional considerations that $m + q = 1/2$ and $n + p = 3/2$; hence, equation (1.1) may be written as

$$\frac{\partial}{\partial t} \int_0^k F(z, t)\, dz = -2\nu \int_0^k z^2 F(z, t)\, dz$$

$$- \gamma \int_k^{\infty} x^m F^n(x, t)\, dx \int_0^k y^{\frac{1}{2} - m} F^{\frac{3}{2} - n}(y, t)\, dy \qquad (2.3)$$

At this point, while certain deductions about the general nature of $F(k, t)$ can be made, any explicit deductions will depend on the particular values that are assigned to m and n. Additionally, a very interesting case, the universal equilibrium, is obtained by replacing the left-hand side of equation (2.3) by $-\epsilon$ and

this case may be treated by considering

$$\epsilon = 2\nu \int_0^k z^2 F(z)\, dz + \gamma \int_k^\infty x^m F^n(x)\, dx \int_0^k y^{\frac{1}{2} - m} F^{\frac{3}{2} - n}(y)\, dy \qquad (2.4)$$

wherein the temporal dependence has been suppressed.

Two sets of values for m and n have received extended treatments, and the purpose of this note is to treat another set.

3. Particular Cases of the Universal Equilibrium Spectrum

In equation (2.4) one observes that the integral

$$\int_0^k y^{\frac{1}{2} - m} F^{\frac{3}{2} - n}(y)\, dy$$

represents the gain of energy from the macro-components of the turbulence and that the integral

$$\int_k^\infty x^m F^n(x)\, dx$$

represents the loss of energy to the micro-components while the integral

$$2\nu \int_0^k z^2 F(z)\, dz$$

is the expression for the viscous dissipation.

Following the trail which was broken by the mixing length theory of Prandtl (Reference [3]), these observations led von Weizsacker (Reference [4]) to postulate that the turbulent cascade is similar to the conversion of mechanical energy into thermal energy by means of molecular motion by taking the micro-components to be an effective viscosity which acts on the vorticity of the macro-components. The von Weizsacker idea was formulated analytically by Heisenberg (Reference [5]), and an explicit solution for the universal equilibrium spectrum was obtained by Bass (Reference [6]). This frequently discussed case is given in the von Kármán transfer function when m and n are assigned respectively the values $-3/2$ and $1/2$.

Another case which is often discussed is one due to Obukhoff (Reference [7]), who postulated an energy transfer because of the strain produced in the macro-components by the Reynolds' stresses in the micro-components; an explicit solution for the equilibrium spectrum for the Obukhoff postulate was obtained by Millsaps (Reference [8]).

Yet another case which should be treated is the self-consistent one which may be deduced from the consideration that the same mechanism which puts energy into an eddy also takes it out. This novel case is given in the von Kármán transfer function by putting $m = 1/4$ and $n = 3/4$, and an explicit solution may be obtained because of some unexpectedly fortuitous algebra.

The integral equation which is to be solved is

$$\epsilon = 2\nu \int_0^k z^2 F(z)\, dz + \gamma \int_k^\infty x^{\frac{1}{4}} F^{\frac{3}{4}}(x)\, dx \int_0^k y^{\frac{1}{4}} F^{\frac{3}{4}}(y)\, dy$$

Two differentiations with respect to k yield the differential equation

$$\nu \left(k^{-1} F + \frac{dF}{dk} \right) = 4\gamma k^{-\frac{3}{2}} F^{\frac{3}{2}}$$

which has the easily deduced solution

$$F(k) = \frac{4\nu^2 k}{(c k^4 + \gamma)^2}$$

wherein c is an arbitrary constant. It is interesting to observe that

$$F(0) = F(\infty) = 0$$

thereby avoiding the infrared and ultraviolet catastrophes and that a cutoff value

at which the spectrum falls discontinuously to zero is also not required. The constant c may be determined from the consideration

$$\epsilon = 2\nu \int_0^{\infty} k^2 F(k)\, dk$$

Thence, one finally has

$$F(k) = \frac{4\gamma^2 \epsilon^2 \nu^2 k}{(2\nu^3 k^4 + \epsilon \gamma^2)^2}$$

for the self-consistent universal equilibrium spectrum.

4. Conclusion

A closed solution for the self-consistent universal equilibrium spectrum of homogeneous-isotropic turbulence has been derived, and the author sends this short derivation as a contribution by an applied mechanist to an anniversary volume to be published on the occasion of the retirement of a distinguished statistician, Paul R. Rider, his former colleague and his enduring friend.

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NATURAL CONVECTION IN BOTTOM NEIGHBORHOOD OF VERTICAL CYLINDERS

M. G. Scherberg

Nomenclature

c_p = Specific heat at constant pressure

k = Thermal conductivity

x_0 = Nondimensionalizing length

$Nu_x = -(\partial T/\partial y)_{y=0}\, x \,/\, (T_w - T_\infty)$ = Local Nusselt Number

$Gr_x = g\,\beta\,(T_w - T_\infty)\, x^3 / \nu$ = Local Grashof Number

β = Coefficient of thermal expansion

T = Absolute temperature

x, y, r = Vertical, horizontal, and radial coordinates with origin at bottom surface point

μ, ν = Absolute and kinematic viscosity, respectively

a = Cylinder radius

g = Acceleration due to gravity

ρ = Fluid density

Pr = Prandtl number

τ = Temperature function

δ = Boundary layer thickness

p = Temperature function coefficient

u_x = Speed function

Subscripts and Superscripts

w = Wall conditions

∞ = Ambient conditions far from wall

$-$ = Nondimensionalized quantity

Summary

Solutions are presented for the natural convection in the bottom neighborhood of large and small radius vertical cylinders under proper nonisothermal boundary conditions in this region. It is shown that the boundary layer appears to diverge in thickness as the bottom is approached. Some interesting differences between large and small cylinders are noted.

1. Introduction

The natural convection from heated or cooled vertical circular cylinders and the vertical plate has received some attention in recent years. The methods generally used are variations on the method introduced by E. Pohlhausen in 1921 (Reference [1]), and that credited to an unpublished paper of Squire by Goldstein. Both these methods give unsatisfactory infinite heat transfer and a boundary layer converging to zero thickness at the bottom of the surface, presumably as a result of the transformations used to convert the partial differential equations defining the flow to ordinary equations. These methods as used have not permitted the imposition of suitable boundary conditions at the bottom of the surfaces. In most cases the vertical temperature gradients at the bottom of the surfaces have not been zero and the downward conduction of heat has not been taken into account in the study. In one case (see Reference [4]) the experimenter found it necessary to introduce an apparent bottom point below the actual plate bottom point to obtain agreement with analytical results obtained by the methods referred to above. It certainly does not appear plausible that the boundary layer should converge to zero while the lateral temperature gradient is not zero.

In order to suitably take into account the bottom boundary conditions, while avoiding for the present the problem of the nature of the downward conduction of heat into the fluid below the surface to a point of zero temperature gradient, it will be assumed that the temperature is variable near the surface bottom in such a way that the vertical temperature gradient is zero at the bottom. The quantities

usually taken as negligible, such as vertical conduction of heat in the boundary layer, the work of the buoyancy forces, and changes in kinetic energy, will also be so treated in this paper. For the present purpose of exploring the nature of the natural convection near the bottom of vertical surfaces, it will suffice to study the vertical circular cylinder in a manner to include the wire and flat plates as special cases.

2. Analysis

The momentum-energy equations will be used. They are, respectively

$$- g \int_a^{a+\delta} (\rho_\infty - \rho) r \, dr + \frac{\partial}{\partial x} \int_a^{a+\delta} \rho \, u^2 \, r \, dr + a\mu \left(\frac{\partial u}{\partial y}\right)_{r=a} = 0 \qquad (1)$$

$$C_p \, g \frac{\partial}{\partial x} \int_a^{a+\delta} \rho \, u \, (T - T_\infty) r \, dr + ak \left(\frac{\partial T}{\partial r}\right)_{r=a} = 0 \qquad (2)$$

A solution in the form of the Squire equations modified by the introduction of a temperature function $\tau(x)$ will be assumed. These are

$$u = u_x \, (\eta^2 - \eta^3) \quad , \quad \eta = 1 - (r - a)/\delta \qquad (3)$$

$$T/T_\infty = 1 + \tau(x) \, \eta^2 \quad , \quad \tau(0) = \tau'(0) = 0 \qquad (4)$$

It will be noted that the function $\tau(x)$ provides the desired bottom temperature conditions. The solution forms (3) and (4) readily enable the reduction of (1) and (2) to ordinary differential equations. These are

$$\frac{d}{dx}\left(a \, \tau \, u_x \, \delta + \frac{2}{7} \tau \, u_x \, \delta^2 \right) - C_2 \, a \, \tau/\delta = 0 \qquad (5)$$

$$\frac{d}{dx}\left(u_x^2\,\delta\,a + \frac{3}{8}u_x^2\,\delta^2\right) - C_3\,\tau(\delta\,a + \delta^2/4) + C_4\,a\,u_x/\delta = 0 \tag{6}$$

in which

$$C_2 = 60k/\rho\,c_p\,g = 60\,\nu/\mathrm{Pr}\,; \quad C_3 = 35g\ ; \quad C_4 = 105\,\nu \tag{7}$$

and the latter term in a derived factor expression $(1/105 - \tau/252)$ was dropped as negligible.

At this stage, the treatment of the cylinder problem takes two directions, one for the larger diameter cylinders which will yield the flat plate solution as diameter increases, and one for the smaller diameters not necessarily related to the flat plate theory.

For the former, the setting of LeFèvre and Ede (Reference [2]) will be used and solutions of the form

$$\begin{aligned} u_x &= u_1 + u_2/a + u_3/a^2 + \cdots \\ \delta &= \delta_1 + \delta_2/a + \delta_3/a^2 + \cdots \end{aligned} \tag{8}$$

in which u_1, u_2, u_3, $\cdots$; δ_1, δ_2, δ_3, $\cdots$ are independent of a will be sought. The solutions will not be carried to terms beyond the powers of $1/a$ greater than the first. For the smaller cylinders, it is clear that solutions of that type would be questionable so that for the smaller cylinders solutions of the form

$$\begin{aligned} u_x &= u_1\,a^{m_1} + u_2\,a^{m_2} + \cdots \\ \delta &= \delta_1\,a^{r_1} + \delta_2\,a^{r_2} + \cdots \end{aligned} \tag{9}$$

in which $m_1, m_2, m_3, \cdots;$ $r_1, r_2, r_3, \cdots$ are non-negative constants, and the u's and δ's as above are independent of a, will be sought.

3. Larger Cylinders

The equations (8) are substituted in (5) and (6) and the terms in the resulting equations collected in powers of $1/a$. Consistent with the assumption in (8) that the u and δ functions are independent of a the solutions sought are those functions which make these coefficients zero. The coefficients of $(1/a)^0$ and $(1/a)^1$ thus yield the following two sets of differential equations.

$$\begin{aligned} &\tau' u_1 \delta_1^2 + \tau u_1' \delta_1^2 + \tau u_1 \delta_1 \delta_1' - C_2 \tau = 0 \\ &2u_1 u_1' \delta_1^2 + u_1^2 \delta_1 \delta_1' - C_3 \tau \delta_1^2 + C_4 u_1 = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} &u_2'(\tau \delta_1^2) + u_2[\tau' \delta_1^2 + \tau \delta_1 \delta_1'] + \delta_2'(\tau u_1 \delta_1) \\ &\qquad + \delta_2[2\tau' u_1 \delta_1 + 2\tau u_1' \delta_1 + \tau u_1 \delta_1'] \\ &\qquad + \frac{2}{7}[\tau' u_1 \delta_1^3 + \tau u_1' \delta_1^3 + 2\tau u_1 \delta_1^2 \delta_1'] = 0 \\ &u_2'(2u_1 \delta_1^2) + u_2[2u_1' \delta_1^2 + 2u_1 \delta_1 \delta_1' + C_4] \\ &\qquad + \delta_2'(u_1^2 \delta_1) + \delta_2[4u_1 u_1' \delta_1 + u_1^2 \delta_1' - 2C_3 \tau \delta_1] \\ &\qquad + \frac{3}{8}[2u_1 u_1' \delta_1^3 + 2u_1^2 \delta_1^2 \delta_1'] - C_3 \tau \delta_1^3/4 = 0 \end{aligned} \tag{11}$$

The equations (10) define the functions u_1 and δ_1. The equations (11) determine the functions u_2 and δ_2 in terms of the u_1, δ_1, determined by (10).

One shows readily that when the cylinder radius (a) approaches ∞ then the equations (5) and (6) become those for the vertical flat plate and that the solutions u_1, δ_1, of equations (10) are also those for the flat plate problem. The solutions u_2, δ_2 of (11) are thus perturbation increments introduced by the cylinder curvature.

Selection of the temperature function τ: It turns out that a temperature function

$$\tau = p\ x^n \quad , \qquad n > 1 \tag{12}$$

will yield particularly simple solutions for both equations (10) and (11) for the bottom region of the surfaces which is of prime interest at this time. The condition $n > 1$ assures the specified bottom boundary conditions $\tau = \tau' = 0$ at $x = 0$. With the selected temperature function equations (10) will have the solution

$$u_1 = a_1\ x^{(1+n)/2} \quad ; \qquad \delta_1 = b_1\ x^{(1-n)/4} \tag{13}$$

in which we see for the first time that the boundary layer thickness* does not approach zero at the bottom of the surface, but on the contrary acquires a rapid growth. Of course, the degree of validity of this result depends, among other things, on the validity of the form of the speed distribution originally assumed, the neglection of the horizontal components of the velocity, and the form of the temperature distribution assumed in the boundary layer. The boundary layer growth near the bottom appears plausible since one might expect the heat to be conducted out further - where the speeds are very low. Good agreement above the bottom regions of the above solutions with those found in Reference [1] by the so-called

*Thickness for vertical flow.

rigorous method increases plausibility of the former solutions. Experimental verification and/or a more precise analysis without the simplifying assumptions are needed.

With the use of equations (10), (12), and (13) one readily finds that

$$a_1 = 4p^{1/2} \sqrt{\frac{35g}{4(3n+5) + 7\,Pr(5n+3)}}$$

$$b_1 = \frac{2\sqrt{15}\;p^{-1/4}}{\sqrt{(5n+3)\,Pr}} \left\{ \frac{4(3n+5) + 7\,Pr(5n+3)}{35g} \right\}^{1/4} \tag{14}$$

4. Cylinder Perturbations

With the solutions (13) found for equations (10) one finds readily that the equations (11) have a solution of the form

$$u_2 = a_2\, x^{(n+3)/4}$$

$$\delta_2 = b_2\, x^{(1-n)/2} \tag{15}$$

in which the coefficients a_2, b_2 are found to be

$$a_2 = a_1\, b_1\, R_1 / 4 \;; \quad b_2 = {b_1}^2\, R_2 / 4$$

$$R_1 = \left\{ \frac{-\,(8n^2/7 + 36n + 188/7) + (235n^2 + 306n + 99)\,Pr/4}{(40n^2 + 136n + 80) + 7(85n^2 + 126n + 45)\,Pr/4} \right\}$$

$$R_2 = - \left\{ \frac{(32n^2/7 + 16n/7 + 16/7) + (180n^2 + 288n + 108)\ Pr/4}{(40n^2 + 136n + 80) + 7(85n^2 + 126n + 45)\ Pr/4} \right\}$$

Thus, the solution (8), up to powers of $(1/a)$ to the first, may now be written in the simple form

$$u_x = u_1(1 + \delta_1\ R_1 / 4a)$$
$$\delta = \delta_1(1 + \delta_1\ R_2 / 4a) \qquad (16)$$

For Prandtl number (.7) and an n range of 1 to 3, the values of $R_1/4$ and $R_2/4$ will run from .021 to .044 and -.046 to -.054, respectively, and, hence, the cylinder perturbations are quite small and free convection from the non-small cylinder $(\delta/a << 1)$ will not differ much from that in flat plates. It is interesting to note that the cylinder effect increases as the boundary layer thickens. Since R_2 is negative and R_1 positive, the cylinder effect decreases the thickness and increases the speed from the corresponding values in the flat plate. Since

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} \Big/ T_\infty = - \frac{2\tau}{\delta} \qquad (17)$$

it follows that there will be slight increases in heat transfer corresponding to the decreases in δ.

5. Small Cylinder Solutions

Since both u and δ must approach zero with the cylinder radius a (the cylinder surface area approaches zero) it follows that a solution of the type (9) with suitable exponents to include the viscous and heat conducting terms in the first-order solutions must be sought. Here, also, only the two lowest-power terms

in (a) will be studied. It turns out that we must seek a solution of the form

$$u_x = u_1 + a^{2/3} u_2 + a^{4/3} u_3 + \cdots$$

$$\delta = a^{1/3} \delta_1 + a \delta_2 + a^{5/3} \delta_3 + \cdots \tag{18}$$

We put equations (18) into (5) and (6) and collect terms in powers of a, $a^{5/3}$, $a^{7/3}$. On setting the coefficients of the two lowest powers equal to zero, we again get two sets of differential equations.

As in the previous treatment the first set defines u_1, δ_1, with the help of which the second set will determine u_2, δ_2.

The sets of equations are, respectively

$$\tau' u_1 \delta_1^3 + \tau u_1' \delta_1^3 + 2\tau u_1 \delta_1^2 \delta_1' - 3.5C_2 \tau = 0$$

$$3u_1 u_1' \delta_1^3 + 3u_1^2 \delta_1^2 \delta_1' - C_3 \tau \delta_1^3 + 4C_4 u_1 = 0 \tag{19}$$

$$u_2'\left[\frac{2}{7} \tau \delta_1^3\right] + \frac{2}{7} u_2 \left[\tau' \delta_1^3 + 2\tau \delta_1^2 \delta_1'\right]$$

$$+ \delta_2' \left[\frac{4}{7} \tau u_1 \delta_1^2\right] + \frac{2}{7} \delta_2 \left[3\tau' u_1 \delta_1^2 + 3\tau u_1' \delta_1^2 + 4\tau u_1 \delta_1 \delta_1'\right]$$

$$+ \tau' u_1 \delta_1^2 + \tau u_1' \delta_1^2 + \tau u_1 \delta_1 \delta_1' = 0 \tag{20}$$

$$u_2' \left(\frac{3}{4} u_1 \delta_1^3\right) + u_2 \left[\frac{3}{4} u_1' \delta_1^3 + \frac{3}{2} u_1 \delta_1^2 \delta_1' + C_4\right]$$

$$+ \delta_2' \left[\frac{3}{4} \delta_1^2 u_1^2\right] + \delta_2 \left[\frac{3}{2} \delta_1 \delta_1' u_1^2 - \frac{3}{4} C_3 \tau \delta_1^2 + \frac{9}{4} u_1 u_1' \delta_1^2\right]$$

$$+ 2u_1 u_1' \delta_1^2 + u_1^2 \delta_1 \delta_1' - C_3 \tau \delta_1^2 = 0$$

One shows readily that equations (19) have a solution of the form

$$u_1 = a_1 x^{(1+n)/2} \quad ; \quad \delta_1 = b_1 x^{(1-n)/6} \tag{21}$$

and equations (20) have a solution of the form

$$u_2 = a_2 x^{(2n+1)/3} \quad ; \quad \delta_2 = b_2 \tag{22}$$

in which the coefficients may be written

$$a_1 = \sqrt{\frac{105g p}{3(n + 2) + Pr(7n + 5)}}$$

$$b_1^3 = \frac{1260 \nu}{Pr(7n + 5)} \sqrt{\frac{3(n + 2) + Pr(7n + 5)}{105g p}}$$

$$a_2 = \frac{a_1}{b_1} R_3 \quad ; \quad b_2 = R_4$$

$$R_3 = \frac{2(86n^2 + 235n + 109) + 4\,Pr(7n + 5)\,(4n + 5)}{3(107n^2 + 230n + 95) + 2\,Pr(7n + 5)\,(47n + 25)}$$

$$R_4 = -\frac{2(2n + 1)\,(113n + 187) + 4\,Pr(2n + 1)\,(84n + 75)}{3(107n^2 + 230n + 95) + 2\,Pr(7n + 5)\,(47n + 25)}$$

Thus, we have

$$\begin{aligned} u_x &= u_1(1 + a^{2/3} R_3 / \delta_1) \\ \delta &= a^{1/3}\,\delta_1 + a\,R_4 = a^{1/3}\,\delta_1(1 + R_4\,a^{2/3} / \delta_1) \end{aligned} \qquad (23)$$

Since the form of u_1 for the small cylinder treatment is exactly the same as that for the large cylinder treatment, comparisons of the coefficients a_1 of the two cases were of interest. The comparisons are shown in Table 1 for Prandtl numbers (1) and (.7).

Table 1

	Pr = 1			Pr = .7		
n	1	2	3	1	2	3
a_1 (large)	.182	.118	.088	.224		.111
a_1 (small)	.158	.111	.086	.188		.103
$b_1\ p^{1/6}$	2.16	2.07	1.95	2.46	2.26	2.12

It is seen that the differences are not very significant, particularly at the higher values of n.

A preliminary study of the parameters R_3 and R_4 in the n range 1 to 3 and the Pr range .7 to 1 indicated that they may be approximated within ±10 percent and ±3 percent, respectively, by the values .42 and 1.22. Thus, we may write

$$u_x = a_1 \, x^{(1+n)/2} \left(1 + 42a^{2/3} \, x^{(n-1)/6} / b_1\right)$$

$$\delta = a^{1/3} \, b_1 \, x^{(1-n)/6} - 1.22a \tag{24}$$

in which the range of values of b_1 is indicated in Table 1 in terms of the temperature function coefficient p.

The rather interesting result, a negative perturbation independent of the temperature function coefficient p, indicated by equations (23) and (24), should have further study.

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The body of this volume was composed on the Vari-Typer in Bodoni Book Style type face. The mathematical notation was displayed in special engineering mathematical type. The typographical work was done at Holloman Air Force Base in New Mexico by the staff of the Assistant for Technical Publications in the Office of Research Analyses of the Office of Aerospace Research.